Debt Maturity and Lumpy Debt[*]

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Abstract

I develop a dynamic capital structure model in which shareholders determine a firm's leverage ratio, debt maturity, and default strategy. In my model, the firm's debt matures all at once. Therefore, after repaying the principal shareholders own all the firm's cash flows and can pick a new capital structure. The possibility to alter the capital structure at maturity gives shareholders the incentive to issue finite maturity debt and allows me to study firms' joint choice of leverage and debt maturity. I also extend my model by allowing for time-varying capital supply to study time-variation in firms' joint choice of leverage and debt maturity.

Keywords: debt maturity; capital structure; default.

JEL classification: G32; G33; G34.

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In the frictionless financial markets of Modigliani and Miller (1958), firm value is independent of financing decisions, implying that capital structure is irrelevant. The insight that market frictions make financing decisions relevant has spawned a large body of theoretical research, most of which focuses on the choice between equity and debt, that is on firms' leverage ratios. While leverage ratios are important in determining the effects of financing decisions on firm value, the global financial crisis of 2007-2009 has shown that debt maturity was a key driver of default risk and therefore of valuations, see Gopalan et al. (2013). In this paper, I develop the first dynamic model of the joint determination of optimal leverage and optimal maturity. I then use this model to produce a new set of testable implications on firms' debt choices and their relation to default risk.

Specifically, I consider a dynamic capital structure model in which firms can choose not only how much debt to issue but also the maturity of their debt. This model builds on the seminal paper of Leland (1994), in which leverage ratios balance the tax benefits of debt with its default costs, but relaxes the assumption of infinite maturity debt. The infinite maturity debt assumption of Leland implies that debt pays a fixed coupon payment over the infinite future, or until the firm defaults. While the assumption of infinite maturity has been used productively over the years, it clearly prevents the analysis of optimal debt maturity. Relaxing this assumption is challenging however, as this implies that when deciding on the firm's debt policy, shareholders need to anticipate their future debt issuance and default decisions.

In my model, shareholders decide on the firm's debt issuance and default strategy. The firm faces three frictions: taxes, that can be lowered by issuing debt; default costs; and debt issuance costs. The firm's debt matures all at once and therefore the maturity structure is *lumpy*. In many of the existing finite debt maturity models the maturity structure is perfectly *granular*, see for example Leland (1998) and He and Xiong (2012b). In my model, when the debt matures the firm repays the principal, then shareholders own all the firm's cash flows and decide on the amount and maturity of a new debt issue. The option to change the capital structure at maturity dates combined with the fact that the firm grows over time gives shareholders the incentive to issue finite maturity debt. This model allows me to study shareholders joint optimal choice of leverage and debt maturity. Economically, the option to relever can be interpreted as financial flexibility, which according to Graham and Leary (2011) survey is one of the key concerns of CFOs. The finite optimal debt maturity result is robust to allowing shareholders to restructure their debt, assuming there are modest restructuring costs. From this model description it becomes clear that shareholders trade-off four factors when deciding on the firm's capital structure: tax benefits of debt, option to relever at maturity, issuance costs, and defaults costs.

To study time-variation in the firm's optimal leverage and debt maturity, the model is extended by allowing for time-varying capital supply. Capital supply is known to influence capital structure, see Faulkender and Petersen (2006), Leary (2009), and Kisgen (2006). The time-variation in capital supply is incorporated using time-varying issuance costs. In my model, there is no commitment by shareholders to an issuance strategy. Each time the firm issues debt, shareholders pick the leverage ratio and debt maturity that maximizes the value of their claim. Because the debt issuance costs vary over time shareholders optimal capital structure varies over time.

In the continuous debt rollover models of Leland (1998) and He and Xiong (2012b), the firm issues a continuum of bonds. When an independent identically distributed Poisson shock hits one of these bonds, it matures. The intensity of the Poisson process determines the debt maturity. Since there is a continuum of bonds, the fraction of bonds that matures each period is deterministic. This setup generates the granular maturity structure in their models. In my model the firm also issues a continuum of bonds with Poisson maturity dates. Only now the maturity dates are perfectly correlated instead of independent.[1] The perfect correlation implies that the debt comes due all at once and leads to the lumpy maturity structure. This setup makes the equity and debt value time homogenous, while still having a firm that issues finite maturity debt.[2]

The model with fixed issuance costs and the model with time-varying issuance costs allow me to study both cross-sectional and time-series variation in the optimal capital structure. First, I examine the cross-sectional implications in the setting with fixed issuance costs. My model is able to replicate the positive relation between leverage and debt maturity that Barclay and Smith (1995), Stohs and Mauer (1996), Johnson (2003), and Custódio et al. (2013) find. The intuition is as follows. Over time the firm grows and therefore over time shareholders want to increase leverage to shield the increasing operating income from taxes. An increase in debt maturity implies that it takes longer before shareholders can increase leverage because shareholders only increase leverage at maturity dates. With a longer debt maturity shareholders anticipate that it takes longer before they can increase leverage. Therefore, shareholders increase leverage ex ante to anticipate the increase in operating income over time.

A comparative statics analysis is performed. Given empirically realistic parameter values, firm value increases by 7.8% because of the net benefits of debt, optimal leverage is 24.8%, and optimal expected debt maturity 9.7 years. These estimates are in line with what is

[1] A different interpretation is that in my model the firm issues a single bond with a Poisson maturity date.

[2] Carr (1998) uses a similar approach to price American options. He uses the Erlang distribution, which is an aggregated exponential distribution, instead of the exponential distribution to model the random maturity date.

empirically observed. Compared to the Leland (1994) model, a special case of my model when the firm issues perpetual debt, firm value increases by 4.5%. From the analysis it becomes clear that debt maturity is often non-monotonic in the model parameters. The reason is that shareholders jointly determine leverage and debt maturity. These non-monotonicities should be taken into account when examining the determinants of capital structure. Most of the comparative statics results for the leverage ratio are in line with Leland (1994). Firms with more volatile cash flows have lower leverage and a longer debt maturity compared to firms with less volatile cash flows. Firms facing higher debt issuance costs have a slightly higher leverage ratio and longer debt maturity. Higher default costs lead to a lower leverage ratio and shorter debt maturity. Finally, firms with higher tax benefits have a higher leverage ratio and most of the time a shorter debt maturity. The effect of changes in tax benefits on debt maturity coincides with the findings of Stohs and Mauer (1996). The comparative statics results on leverage choice are in accordance with results found in Harris and Raviv (1991), Frank and Goyal (2009), Rajan and Zingales (1995), Graham (1996a), and Graham et al. (1998).

The model with time-varying capital supply allows me to study time-series implications. Given realistic parameter values, for high issuance costs, compared to low issuance costs, shareholders issue less debt with a longer maturity. The higher issuance costs cause issuing debt to be less profitable and it takes a longer time to recuperate the issuance costs, leading to the time variation in optimal capital structure. The results are economically significant, with a 7 month difference in maturity (9.6 years versus 9 years) and a 15% difference in leverage (22% versus 25.5%) between the high and low issuance costs state.

This article merges two strands of the literature. First, it relates to models with finite debt maturity based on the classical work by Black and Scholes (1973) and Merton (1974). Second, it relates to dynamic capital structure models based on Leland (1994) with finite maturity debt, see for example Leland and Toft (1996) and Leland (1998). Economists merged these two strands of the literature before. The difference between the existing literature and my model is that the default barrier is endogenous, while in Ju et al. (2005) and Ju and Ou-Yang (2006) the default barrier is exogenous, and I show optimality and existence of a solution, while Flor and Lester (2002) and Childs et al. (2005) solve their models only numerically.

My article also builds on and contributes to other strands of the literature. Much work has been done on the risks coming from short-term debt, see He and Xiong (2012a), Cheng and Milbradt (2012), Brunnermeier and Oehmke (2013), He and Milbradt (2014a), He and Xiong (2012b), He and Milbradt (2014b), and Della Seta et al. (2015). I also look at these risks but my work differs from the existing literature either in the way it incorporates finite maturity debt and/or the fact that shareholders do not commit to an issuance strategy. My model incorporates time-varying debt issuance costs in the same way Hackbarth et al. (2006), Bolton et al. (2013), Bhamra et al. (2010), Chen et al. (2013), and Chen et al. (2014) incorporate macroeconomic states of the economy. The different way in which I model finite maturity debt leads to a time-varying optimal debt maturity (and leverage) without commitment. In Appendix A, I extend my model by allowing for debt restructuring. To do this I build on work by Fischer et al. (1989), Goldstein et al. (2001), Strebulaev (2007), and Hugonnier et al. (2015). The lumpy maturity structure implies that a barrier restructuring strategy, which they find to be optimal, is not always optimal in my case. Finally, as in Décamps and Villeneuve (2012), Dangl and Zechner (2006), and Hugonnier et al. (2015), I study the game between creditors and shareholders.

The next section describes the model and gives its solution. The second section looks at the capital structure choice shareholders make in equilibrium. The third section studies the effects of time-varying capital supply. The final section concludes. The appendix contains an extension of the model that allows for debt restructuring and all the proofs.

I. The Model

To make the analysis transparent I use the framework of Leland (1994) to which I add the choice of debt maturity. The probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$, where $\mathbb{Q}$ is the risk-neutral probability measure, represents the uncertainty in the economy. Agents in the economy discount the cash flows they receive at the constant rate $r > 0$. Shareholders control the capital structure and default decisions of the firm. They receive dividends and debt proceeds from the firm until it defaults.

The firm has assets in place with a value of V_t. This asset value V_t evolves over time according to a geometric Brownian motion,

$$dV_t = (r - \delta)V_t dt + \sigma V_t dB_t,$$

with B_t a Brownian motion under the risk-neutral probability measure $\mathbb{Q}$, $r - \delta$ the drift, and $\sigma > 0$ the volatility. Each instant a fraction $\delta > 0$, the payout rate, of the value of the assets in place is paid out to shareholders as after tax operating income $\delta V_t > 0$. This implies that the firm's operating income is independent of its financing decisions. Furthermore, all expected discounted operating income is equal to V_t,

$$V_t = \mathbb{E}_0 \left[\int_t^{\infty} e^{-r(s-t)} \delta V_s ds \right].$$

The value of the assets in place V_t is also known as the unlevered firm value.

The operating income can be shielded from taxes by issuing debt. The debt contract is characterized by its coupon C_0 and maturity m_0. The tax benefits are a fraction $\pi \in (0,1)$ of the coupon and come with a full loss offset provision. The debt has a principal $\rho(m_0)C_0$, which depends on the coupon and on the maturity of the debt contract. At maturity τ^1_{mat} the principal needs to be repaid. In a later section $\rho(m)$ is set such that debt is issued at par. If the debt matures shareholders repay the principal and there is no more debt outstanding. At this point in time shareholders again own all cash flows from the firm and have the possibility to issue new debt. They issue new debt with a coupon $C_{\tau^1_{mat}}$ and maturity $m_{\tau^1_{mat}}$. This implies that the process describing the coupon C_t and debt maturity m_t is piecewise constant over time and only changes at maturity dates τ^i_{mat} with $i \in \mathbb{N}$. At these dates debt comes due, shareholders repay the principal and issue new debt. This process is depicted in Fig. 1, at the first maturity date τ^1_{mat} only the coupon of the new debt issue changes while the debt maturity stays the same. In the figure, at the second maturity date τ^2_{mat} both the coupon and the maturity of the debt change.

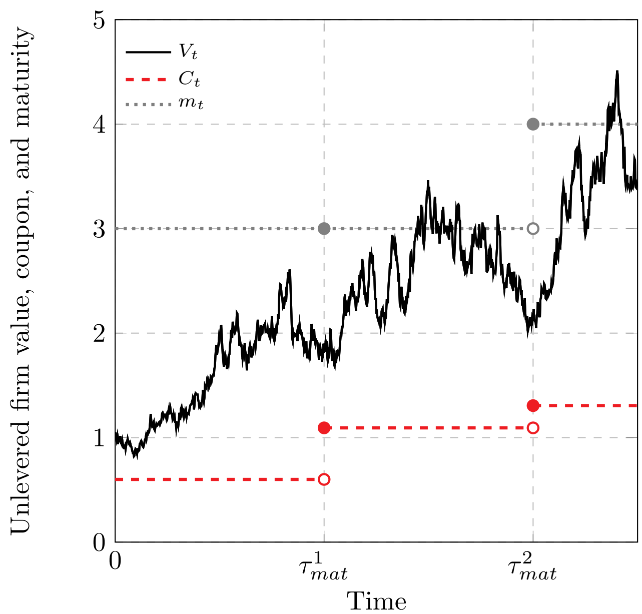


FIGURE 1: The unlevered firm value V_t, coupon C_t, and debt maturity m_t over time. The coupon and debt maturity are piecewise constant and only change at the maturity dates τ^i_{mat}.

The debt matures when a Poisson process, with intensity $\eta_0 = 1/m_0$, jumps for the first time, which happens at τ^1_{mat}. The expected maturity of the debt ignores the possibility of default and is therefore $\mathbb{E}_0[\tau^1_{mat}] = m_0$. Similarly, the time the second batch of debt is outstanding $\tau^2_{mat} - \tau^1_{mat}$ is given by a Poisson process with intensity $\eta_{\tau^1_{mat}} = 1/m_{\tau^1_{mat}}$ and therefore $\mathbb{E}_{\tau^1_{mat}}[\tau^2_{mat} - \tau^1_{mat}] = m_{\tau^1_{mat}}$. This setup is similar to Leland and Toft (1996) and Leland (1998). In these models there is a continuum of bonds with Poisson maturity dates that are independent and therefore each instant a fixed fraction of the bonds matures. In my model there is also a continuum of bonds with Poisson maturity dates but the maturity dates are perfectly correlated.

Shareholders default at τ_D, a stopping time with respect to $\mathcal{F}$. In case of default creditors take over the firm and the bankruptcy proceeds are $(1-\alpha)V_{\tau_D}$. In default, a fraction α of the unlevered firm value is lost as a frictional cost. Debt value is therefore defined by,

$$\begin{aligned} D(V_0, C_0, m_0|\tau_D) =& \mathbb{E}_0\left[\int_0^{\tau_D \wedge \tau^1_{mat}} e^{-rt} C_t dt\right] \\ &+ \mathbb{E}_0\left[\mathbb{1}_{\left\{\tau^1_{mat} < \tau_D\right\}} e^{-r\tau^1_{mat}} \rho\left(m_{\tau^{1-}_{mat}}\right) C_{\tau^{1-}_{mat}}\right] \\ &+ \mathbb{E}_0\left[\mathbb{1}_{\left\{\tau_D \le \tau^1_{mat}\right\}} e^{-r\tau_D}(1-\alpha)V_{\tau_D}\right]. \end{aligned} \tag{1}$$

Equation (1) shows that creditors receive coupon payments C_t until either shareholders default or the debt matures (first term). If the debt matures before the firm defaults creditors get the principal $\rho(m_{\tau^{1-}_{mat}})C_{\tau^{1-}_{mat}}$ [3] back (second term). Otherwise, creditors get the liquidation value of assets $(1-\alpha)V_{\tau_D}$ (third term). The debt value depends on the unlevered firm value V, coupon C, debt maturity m, and default time τ_D.

In the following, it will be useful to define Z_t as the ratio of the coupon payments C_t over the unlevered firm value V_t, i.e. $Z_t = C_t/V_t$. The issuance strategy $\theta = (Z^\theta_t, m^\theta_t)_{t\ge 0}$ determines the choice of coupon and maturity when the firm issues new debt. The fact that coupon and maturity are non-negative implies that Z^θ_t and m^θ_t are restricted to be non-negative. Given an issuance strategy θ if at time t debt is issued then the firm issues debt with coupon $Z^\theta_t V_t$ and maturity m^θ_t. Denote the value of equity by $E(V, C, m|\theta, \tau_D)$. As the debt value, it depends on the three state variables and the default time but it also depends on the debt issuance strategy θ.

Given proportional issuance costs of debt $q \in (0, \pi)$, shareholders claim at the time of debt issuance is,

$$\circleddash(V_t|\theta, \tau_D) = E(V_t, Z^\theta_t V_t, m^\theta_t|\theta, \tau_D) + (1-q)D(V_t, Z^\theta_t V_t, m^\theta_t|\tau_D), \tag{2}$$

which is the equity value plus the proceeds of the debt issue given the initial unlevered firm value, issuance strategy, and default strategy.

Fig. 2 describes the model graphically, with i the number of times the firm has already issued debt. At the top, shareholders issue debt for the $i+1$ time and obtain the net proceeds

[3]The coupon payment C_{t-} is the left-sided limit of the coupon payment when approaching time t, $C_{t-} = \lim_{s\uparrow t} C_s$.

$(1-q)D(V_{\tau^i_{mat}}, C_{\tau^i_{mat}}, m_{\tau^i_{mat}}|\tau_D)$ while creditors buy the debt for $D(V_{\tau^i_{mat}}, C_{\tau^i_{mat}}, m_{\tau^i_{mat}}|\tau_D)$. After debt issuance, the firm has issued debt $i = i+1$ times, shareholders claim is given by the equity value $E(V_t, C_t, m_t|\theta, \tau_D)$, and creditors claim by the debt value $D(V_t, C_t, m_t|\tau_D)$. The firm makes coupon payments to creditors C_t and dividend payments $\delta V_t - (1-\pi)C_t$, operating income minus coupon payments corrected for tax benefits, to shareholders. If the dividends are negative deep pocketed shareholders invest in the firm. Two things can then happen: either the debt matures or the firm defaults. If the firm defaults, shareholders abandon their claim and the equity value is zero because of limited liability while creditors get the bankruptcy proceeds $(1-\alpha)V_{\tau_D}$. If the debt matures, shareholders decide whether to default or repay the principal on outstanding debt. In the first case, the cash flows and value of the claims are as before. In the second case, the equity value is $\mathcal{O}(V_{\tau^i_{mat}}|\theta, \tau_D) - \rho(m_{\tau^{i-}_{mat}})C_{\tau^{i-}_{mat}}$, the firm value after all the debt is paid off $\mathcal{O}(V_{\tau^i_{mat}}|\theta, \tau_D)$ minus the principal that needs to be repaid to creditors $\rho(m_{\tau^{i-}_{mat}})C_{\tau^{i-}_{mat}}$. Creditors claim is the principal $\rho(m_{\tau^{i-}_{mat}})C_{\tau^{i-}_{mat}}$. After the principal is repaid, shareholders fully own the firm and can pick a new capital structure while creditors have no claim on the firm's cash flows. This implies that shareholders value is $\mathcal{O}(V_{\tau^i_{mat}}|\theta, \tau_D)$ while creditors value is zero. The shareholders then again issue debt and the process repeats itself.

This implies that the equity value is defined by,

$$\begin{aligned} E(V_0, C_0, m_0|\theta, \tau_D) =& \mathbb{E}_0\left[\int_0^{\tau_D \wedge \tau^1_{mat}} e^{-rt}\left(\delta V_t - (1-\pi)C_t\right)dt\right] \\ &+ \mathbb{E}_0\left[\mathbb{1}_{\left\{\tau^1_{mat} < \tau_D\right\}} e^{-r\tau^1_{mat}}\left(\mathcal{O}\left(V_{\tau^1_{mat}}|\theta, \tau_D\right) - \rho\left(m_{\tau^{1-}_{mat}}\right)C_{\tau^{1-}_{mat}}\right)\right]. \end{aligned} \quad (3)$$

Shareholders receive the operating income minus the after tax coupon payments from the firm $\delta V_t - (1-\pi)C_t$ until either the firm defaults τ_D or the debt matures τ^1_{mat} (first term). When the debt matures before the firm defaults, shareholders repay the principal and continue to operate the firm $\mathcal{O}(V_{\tau^1_{mat}}|\theta, \tau_D) - \rho(m_{\tau^{1-}_{mat}})C_{\tau^{1-}_{mat}}$ (second term). Otherwise, the payoff is zero.

A. Shareholders versus Creditors

The model spelled out contains a rational expectations problem, to be more precise it contains an infinitely repeated game between strategic shareholders and competitive creditors. The fact that creditors are competitive implies that in equilibrium the debt price is its expected discounted cash flows. For creditors to calculate the expected discounted cash flows, they need to conjecture the default strategy used by shareholders. In case shareholders deviate from this default strategy, either shareholders or creditors lose money, since the debt value ex ante differs from the expected discounted cash flows given the ex post behavior of shareholders.

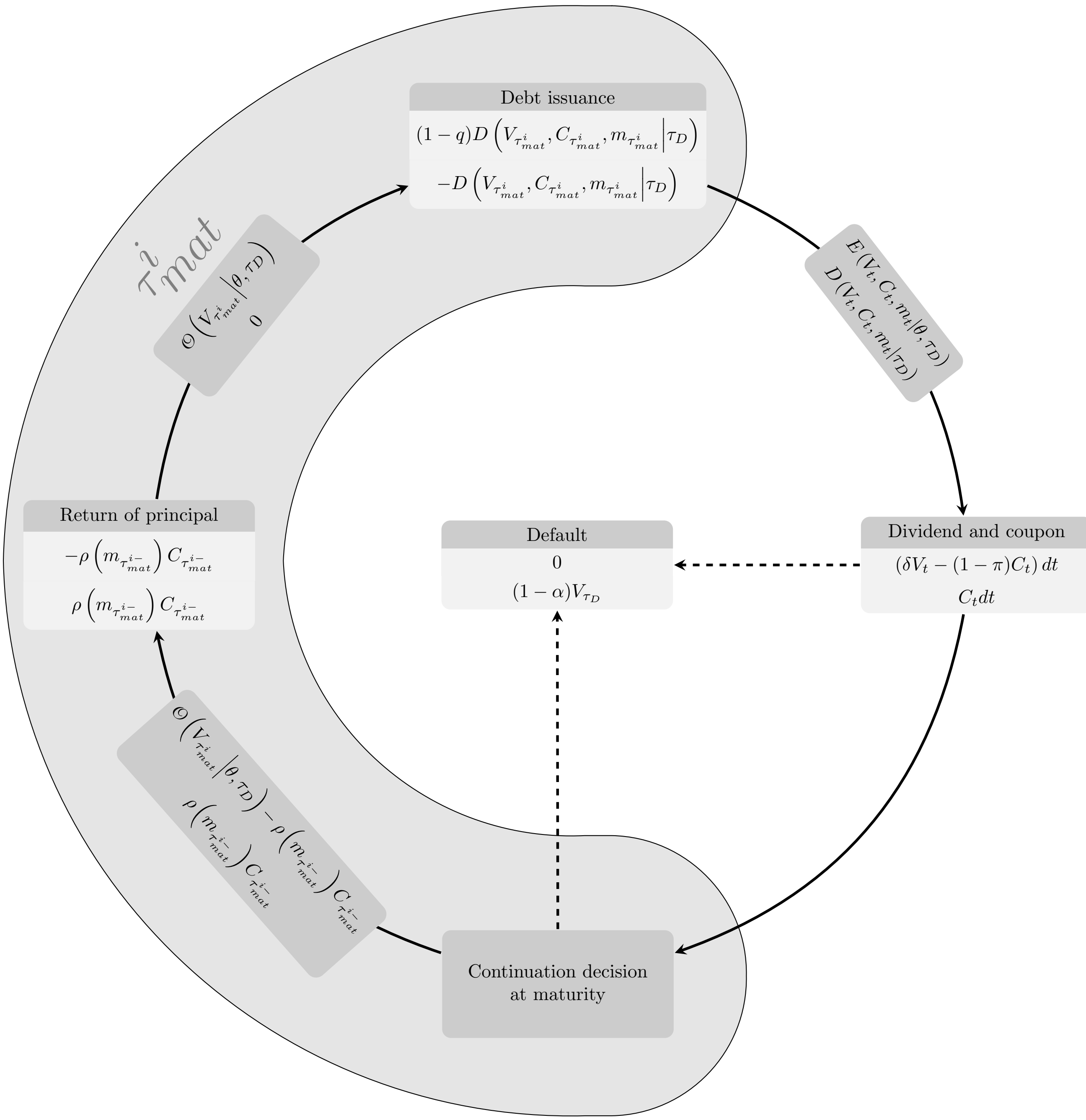


FIGURE 2: The cash flows (the two toned boxes) to shareholders and creditors and the value of shareholders and creditors claim (single toned boxes). The top equation is for shareholders and the bottom equation for creditors. The gray shaded area depicts the maturity date τ^i_{mat}.

The repeated game continues until shareholders decide to default, after that creditors own the firm. The firm and equity value at maturity as defined in equation (2) and (3) already use the equilibrium price of the debt.

The constituent game is from the moment the firm issues debt until it repays the principal or defaults. Each constituent game has the following steps:

1. Shareholders issue debt with coupon $Z_0^\theta V_0$ and maturity m_0^θ.

2. Competetive creditors buy the debt for a price $P(V_0, Z_0^\theta V_0, m_0^\theta)$.

3. Shareholders determine a default strategy and the firm pays coupon and dividends until the debt matures or the firm defaults $\tau_{mat}^1 \wedge \tau_D$.

4. If the debt matures before the firm defaults $\tau_{mat}^1 < \tau_D$ shareholders return the principal $\rho(m_{\tau_{mat}^{1-}}) C_{\tau_{mat}^{1-}}$ and go to step 1. Otherwise, at τ_D shareholders default, the bankruptcy proceeds $(1-\alpha)V_{\tau_D}$ go to creditors, and the game ends.

Shareholders strategy in one constituent game exists of picking a coupon Z_0^θ, debt maturity m_0^θ, and default strategy τ_D. Creditors strategy is setting a price $P(\cdot,\cdot,\cdot)$ for any debt issuance strategy (Z_0^θ, m_0^θ) given V_0.

This paper looks for a Markov perfect equilibrium defined as follows:[4]

Definition 1. *A Markov perfect equilibrium is a debt issuance strategy θ, default strategy τ_D, and debt price $P(\cdot,\cdot,\cdot)$ that are Markovian such that:*

1. *Shareholders maximize the equity value plus the net debt proceeds at maturity dates and the equity value at any other date.*

2. *Creditors make zero profit in expectation on each debt issue.*

These requirements imply that in equilibrium,

$$P(V_t, Z_t^\theta V_t, m_t^\theta) = D(V_t, Z_t^\theta V_t, m_t^\theta | \tau_D).$$

The firm value in equation (2) and the equity value in equation (3) already incorporate this equilibrium condition. In this equilibrium, when shareholders issue debt they maximize the equity value plus the net proceeds from the debt issue. Thereafter, shareholders choose the default time to maximize the equity value. Creditors conjecture shareholders behavior correctly and the debt price is its expected discounted cash flows. Furthermore, both shareholders and creditors have no incentive to deviate from their strategy.

Because all payoffs are homogeneous in V and C it seems natural that an issuance strategy should be homogeneous in V as described below:

Definition 2. *A homogeneous and Markovian issuance strategy is characterized by two constants $\theta_M = (Z^{\theta_M}, m^{\theta_M}) \in \mathbb{R}_2^+$ such that, at maturity, shareholders issue debt with coupon $Z^{\theta_M} V$ and maturity m^{θ_M}.*

[4]For further information on Markov perfect equilibria see Maskin and Tirole (2001).

B. Model Solution

The homogeneity of the model in V and C implies that for a homogeneous and Markovian issuance strategy θ_M and appropriate default strategy τ_D the equity and debt value can be rewritten as,

$$E(V, C, m|\theta_M, \tau_D) = Ve(Z, m|\theta_M, \tau_D),$$
$$D(V, C, m|\tau_D) = Vd(Z, m|\tau_D).$$

Fix a homogeneous and Markovian debt issuance strategy θ_M. If at time t the debt does not mature shareholders claim is equal to the equity value and the optimal default strategy is given by a barrier default strategy in Z. When the debt matures shareholders only continue to operate the firm when the value of their claim is positive,

$$\mathcal{O}(V_t|\theta_M, \tau_D) - \rho(m_{t^-})C_{t^-} \geq 0.$$

This implies that there are two types of default, as can also be seen from Fig. 2. The first type occurs during normal operations when the operating income of the firm deteriorates sufficiently and shareholders stop making coupon payments. I call this a coupon default $\mathcal{C}$. The second type of default occurs at maturity when the principal comes due. If this principal is too large relative to the expected future income of the firm, shareholders prefer to abandon their claim. I call this a principal default $\mathcal{P}$. The optimal default time is given by,

$$\begin{aligned} \mathcal{C} &= \inf\left\{t \geq 0 | Z_t \geq Z_D \text{ s.t. } \forall i \in \mathbb{N}\ t \neq \tau^i_{mat}\right\}, \\ \mathcal{P} &= \inf\left\{t \geq 0 | \mathcal{O}(V_t|\theta_M, \tau_D) - \rho(m_{t^-})C_{t^-} < 0, \exists i \in \mathbb{N} \text{ s.t. } t = \tau^i_{mat}\right\}, \\ \tau_D &= \inf\left\{\mathcal{C}, \mathcal{P}\right\}. \end{aligned} \tag{4}$$

For any issuance strategy given a fixed default strategy τ_D a contraction mapping argument shows that a solution to the model exists. For tax benefits smaller than $\pi^*(\pi)$ defined as,

$$\pi^*(\pi) = \min_{\eta \in [0, (\pi - q)/\rho(1/\eta)q]} \frac{1 + \eta\rho(1/\eta) + \eta\frac{(q + q\eta\rho(1/\eta))}{\delta}}{1 + \frac{\eta}{\delta}},$$

there also exists a finite optimal firm value and default threshold.

The intuition behind this restriction is as follows. For short-term debt, maturities $m = 1/\eta$ with $\eta > (\pi - q)/\rho(\eta)q$, the firm value is always finite. The fact that the debt needs to be rolled over frequently combined with the fact that issuance costs are paid over the principal while no tax benefits are received on the repayment of the principal guarantee existence of a finite firm value. For debt with longer maturities, maturities $m = 1/\eta$ with $\eta < (\pi - q)/\rho(\eta)q$, this effect is no longer sufficient to guarantee existence of a finite firm value. Therefore, the tax benefits should be small enough such that the firm value is finite and shareholders find it optimal to default at some point. For debt with maturity $m = 1/\eta$ the restriction $\pi < \frac{1+\eta\rho(1/\eta)+\eta\frac{(q+q\eta\rho(1/\eta))}{\delta}}{1+\frac{\eta}{\delta}}$ ensures this. This restriction leads to $\pi^*(\pi)$. The bound on the tax benefits $\pi^*(\pi)$ is a sufficient condition for the existence of a solution but not a necessary condition.

This leads to the first result:

Theorem 1. *For $\pi < \pi^*(\pi)$ there exists a Markov perfect equilibrium. In this equilibrium,*

- *The equilibrium issuance strategy θ_M^* maximizes the firm value assuming shareholders default optimally.*
- *The equilibrium default strategy τ_D^* is,*

$$\tau_D^* = \inf \{\mathcal{C}, \mathcal{P}\},$$

  where the default barrier Z_D^* satisfies the smooth pasting condition.
- *The equilibrium debt price equals the expected discounted cash flows of the debt given τ_D^*.*
- *Off-equilibrium, creditors belief that any deviation from the optimal issuance strategy is a one-shot deviation. The off-equilibrium debt price is the expected discounted cash flows of the debt given the optimal default strategy that follows from these beliefs.*

The appendix contains all the proofs. If shareholders issue an off-equilibrium coupon and maturity then the debt proceeds are as if this deviation is a one-shot deviation and shareholders optimally default given this one-shot deviation. For example, if shareholders issue debt with an off-equilibrium coupon that is higher and maturity that is longer, then creditors assume that in the future shareholders return to issuing debt with the equilibrium coupon and maturity. Creditors assume that the deviation in the issuance strategy happens only once, a one-shot deviation. Furthermore, they assume that shareholders optimally default given this one-shot deviation in the issuance strategy. Creditors then use this conjectured default strategy to value the debt claim. In the rest of this article the optimal coupon and maturity are abbreviated by $Z^* = Z^{\theta_M^*}$ and $m^* = m^{\theta_M^*}$ where possible.

II. Model Analysis

Having shown the existence of a solution to the model, the next step is to further characterize this solution. First, the optimal debt maturity is examined after which a comparative statics analysis is performed.

A. Debt Maturity

In most existing dynamic capital structure models with finite maturity debt, firms optimally issue perpetual debt to avoid rollover risk. In my setup this is not the case because of the option to relever at maturity. This section formalizes this intuition and shows that the optimal debt maturity is positive and finite.

The next proposition shows that there is a lower bound on the maturity of the debt issued,

Proposition 1. *The equilibrium debt maturity m^* from Theorem 1 is positive $m^* > \bar{m} > 0$, where the constant $\bar{m}$ is defined in the appendix.*

This result has an intuitive explanation. On the one hand, the firm receives tax benefits over the coupon payments but not over the principal repayments. On the other hand, the firm pays issuance costs over both the coupon payments and the principal repayments. The shorter the debt maturity the larger the issuance costs relative to the tax benefits. Furthermore, the shorter the debt maturity the closer the option value of relevering is to zero. The low value of the option to relever for short maturities follows from the fact that the expected operating income at maturity is similar to the operating income now and the variance of the operating income at maturity is small. If shareholders can only issue debt with a maturity shorter than $\bar{m}$ the costs of issuing debt are larger than the benefits and shareholders prefer to abstain from issuing debt. Therefore, if shareholders issue debt the maturity must be longer than $\bar{m}$. Otherwise, issuing debt would decrease shareholder value.

The second proposition obtains the finite debt maturity result in a special case,

Proposition 2. *For small issuance costs q and principals $\rho(m)$ the equilibrium debt maturity m^* from Theorem 1 is finite, that is $m^* < \infty$.*

The option to relever at maturity gives shareholders the incentive to issue finite maturity debt. If they issue perpetual debt, they forgo on the option to relever at maturity. The next subsection shows that this result holds more generally since shareholders trade-off issuance and default costs versus tax benefits and the option to relever the firm. Without the option to relever, the firm would issue perpetual debt to minimize the issuance costs and rollover risk, as in Leland and Toft (1996) and Leland (1998). This proposition makes the intuition of Kane et al. (1985) and Ju and Ou-Yang (2006) rigorous.

The intuition behind relevering is similar to debt restructuring, as in Fischer et al. (1989), Goldstein et al. (2001), and Strebulaev (2007), but there are differences. First, in my setup shareholders can relever both upward and downward at maturity. Restructuring on the other hand can usually only be done upward because of the creditor holdout problem, see Hugonnier et al. (2015). In my model, the debt value at maturity is given by the principal and therefore the holdout problem no longer exists. Furthermore, there are often extra costs related to restructuring debt with dispersed creditors compared to regular debt issuance, see Gilson (1997). Appendix A extends the model by allowing for upward restructuring and shows that for reasonable restructuring costs shareholders still issue finite maturity debt.

B. *Comparative Statics Analysis*

This subsection performs a numerical comparative statics analysis assuming that debt is issued at par. Let $\bar{Z}(m)$ be the firm value maximizing coupon over unlevered firm value given that shareholders only issue debt with maturity m. The principal $\rho(m)$ is determined by solving,

$$\rho(m) = \sup \left\{ \bar{\rho} \in [0, 1/r] \middle| \bar{\rho}\bar{Z}(m)V \leq D(V, V\bar{Z}(m), m|\bar{\theta}_M, \tau_D)|_{\rho=\bar{\rho}} \right\}, \tag{5}$$

with $\bar{\theta}_M = (\bar{Z}(m), m)$. In all the cases examined here the inequality given above is an equality for $\rho(m)$. This implies that for each maturity m and optimal $\bar{Z}(m)$, debt is issued at par.

To perform the numerical comparative statics analysis I use the base case parameters from Table 1. The unlevered firm value V_0 is 1 without loss of generality because of the homogeneity properties of the model. The discount rate is 4.21%, the same as in Morellec et al. (2012), who calibrated the discount rate on the one-year Treasury rate. Morellec et al. (2012) estimate that the risk-neutral volatility of the operating cash flow is 28.86%. They use data from Compustat, CRSP, and Institutional Brokers' Estimate System for the estimation. The payout rate is $\delta = 1\%$. As in Hugonnier et al. (2015), who base their estimate on Graham (1996b), the tax benefits of debt are 15%. Glover (2014) estimates the default costs at 45% of the firm value. Finally, in Altinkilic and Hansen (2000) the mean issuance costs of debt is 1.09% of the gross proceeds but for highly rated firms their estimate is approximately 0.75% and I use this estimate.

Table 2 contains the results for the implied variables. The optimal firm value is 1.078. The firm value increases by 7.8% due to the net benefits of debt. This estimate is slightly higher than Korteweg (2010) and van Binsbergen et al. (2010). The former says that up to

Parameter	Symbol	Value
Unlevered firm value	V_0	1
Discount rate	r	0.042
Payout rate	δ	0.01
Volatility	σ	0.2886
Tax benefits	π	0.15
Default costs	α	0.45
Issuance costs	q	0.0075

Table 1: The base case parameters used in the comparative statics analysis.

5.5% of firm value follows from the net benefits of debt and the later says 3.5%. The leverage ratio,

$$L(V_0) = \frac{D(V_0, Z^*V_0, m^*|\theta_M^*, \tau_D^*)}{E(V_0, Z^*V_0, m^*|\theta_M^*, \tau_D^*) + D(V_0, Z^*V_0, m^*|\theta_M^*, \tau_D^*)},$$

in the base case model is 24.8% and is comparable to Korteweg (2010) estimate of 23.4%. Furthermore, the optimal debt maturity is 9.69 years, which is in the middle of the median maturity Custódio et al. (2013) find for bond issues (13.1 years for 1976-2008) and syndicated loan issues (3.9 years for 1987-2008).

Parameter	Symbol	Base Case	Leland (1994)
Firm value	$\mathcal{O}(V_0\|\theta_M^*, \tau_D^*)$	1.078	1.033
Leverage ratio	$L(V_0)$	0.248	0.384
Debt maturity	m^*	9.694	∞
Principal	$\rho(m^*)$	21.535	19.515
Coupon	Z^*	0.012	0.020
Default threshold	Z_D^*	0.095	0.105

Table 2: The implied variables for the base case model and Leland (1994) model.

A number of empirical studies have documented a positive relation between leverage and debt maturity, see for example Barclay and Smith (1995), Stohs and Mauer (1996), Johnson (2003), and Custódio et al. (2013). To examine the positive relation between leverage and maturity in my model, Fig. 3 fixes the debt maturity but optimizes over the coupon and plots the optimal firm value (left) and leverage ratio (right). As shown by the figure, increasing the debt maturity increases the leverage ratio. The economic intuition is as follows. Issuing debt with a longer maturity increases the time until the shareholders relever the firm. Naturally, this increases the incentive for shareholders to issue debt because they expect the firm value and therefore the operating income to increase over time, while the next date at which the firm can be relevered is deferred. The shorter maturity gives shareholders more financial flexibility, which according to Graham and Leary (2011) survey is one of the key concerns of CFOs

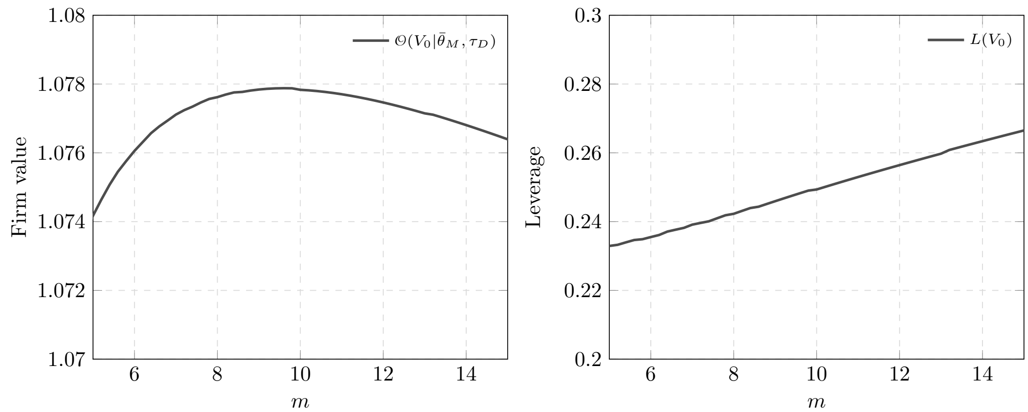


FIGURE 3: The effect of changes in the debt maturity m on the optimal firm value $\mathbb{O}(V_0|\bar{\theta}_M, \tau_D)$ and the optimal leverage $L(V_0)$.

Fig. 4 and Fig. 5 display the comparative statics results for deviations from the base case when shareholders optimize over both the coupon and debt maturity. The left figures show the changes in the optimal firm value $\mathbb{O}(V_0|\theta_M^*, \tau_D^*)$ and leverage ratio $L(V_0)$. The middle figures show the changes in the optimal debt maturity m^* and the right figures the changes in the principal per unit of coupon $\rho(m^*)$. The marked lines in the left figures depict the results for the Leland (1994) model,

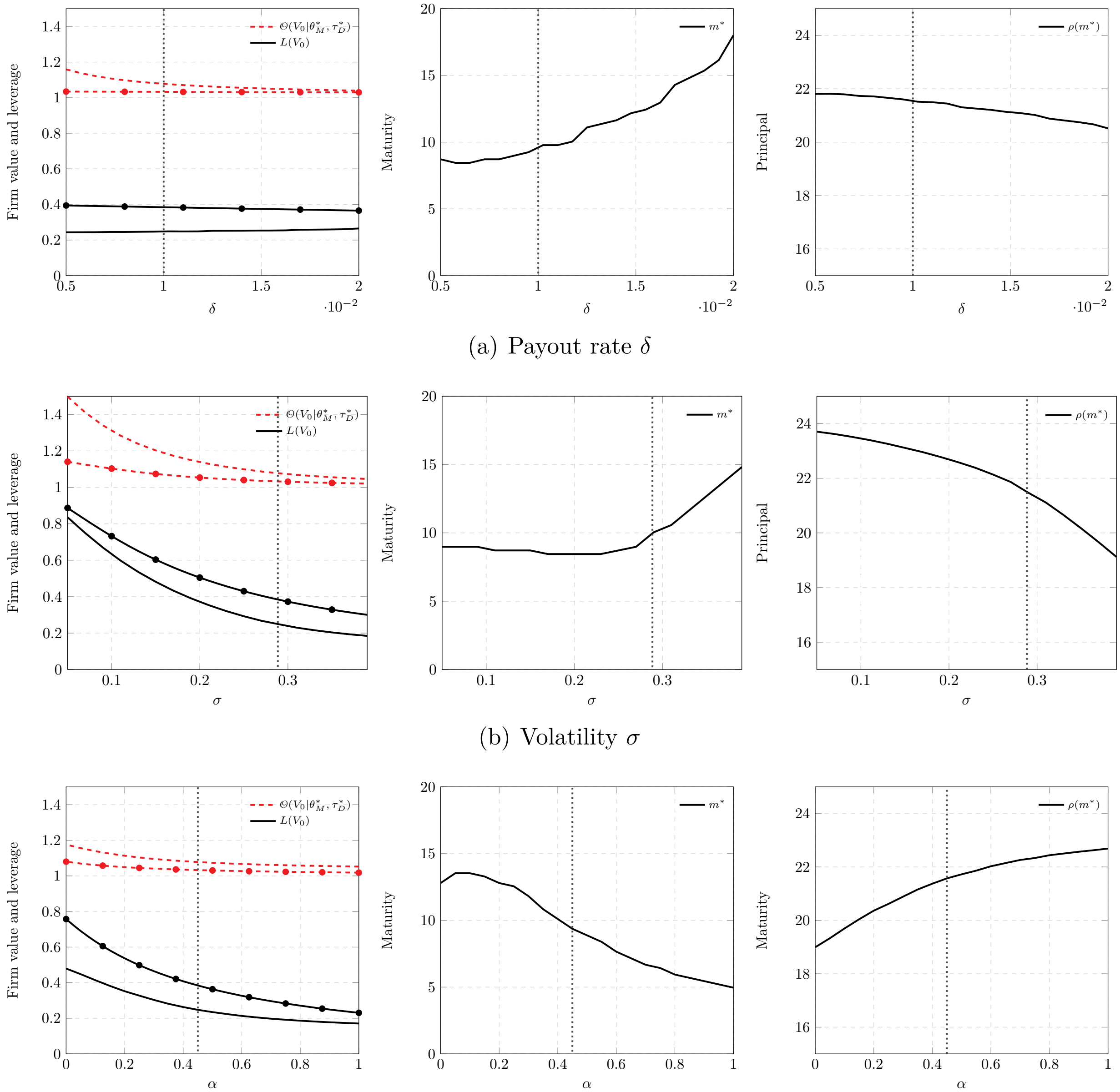
$$\begin{aligned} E(V, C, \infty|\theta_M, \tau_D) =& \mathbb{E}_0\left[\int_0^{\tau_D} e^{-rt}\left(V_t - (1-\pi)C_t\right)dt\right], \\ D(V, C, \infty|\tau_D) =& \mathbb{E}_0\left[\int_0^{\tau_D} e^{-rt}C_t dt + e^{-r\tau_D}(1-\alpha)V_{\tau_D}\right], \end{aligned}$$

given the base case parameters.

The Leland (1994) model is a special case of my model when the firm issues perpetual debt. Observe in Fig. 4, Fig. 5, and Table 1 that because of the option to relever the leverage ratio for the Leland (1994) model is higher than the leverage ratio for my model. This relates to the fact that the firm cannot relever in Leland (1994). Therefore, shareholders opt for a high initial debt level. Table 1 also shows that the possibility for shareholders to choose the debt maturity increases firm value by approximately 4.5%. The comparative statics results for leverage in general coincide with the ones found by Leland (1994), except for the payout rate and issuance costs.

The trade-off between issuance costs, default costs, tax benefits, and the option to relever simultaneously determines the firm's leverage ratio and debt maturity. The effect of debt maturity on leverage and vice versa leads to many of the non-monotonicities in Fig. 4 and

Fig. 5.



(a) Payout rate δ

(b) Volatility σ

(c) Default costs α

FIGURE 4: The effect of changes in the payout rate δ, the volatility σ, and default costs α on the optimal firm value $\Theta(V_0|\theta_M^*, \tau_D^*)$, the optimal leverage $L(V_0)$, the optimal debt maturity m^*, and the principal per unit of coupon $\rho(m^*)$. The marked lines depict the comparative statics results for the Leland (1994) model.

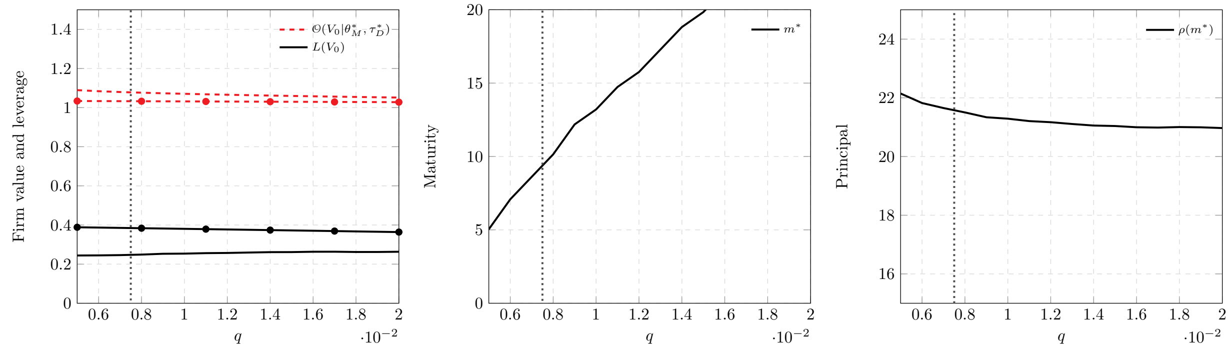
A higher payout rate δ implies that in expectation the growth rate of the firm $r - \delta$ becomes smaller. This has the effect that the value of the option to relever decreases and therefore the firm issues longer term debt, which has as a side effect that the leverage ratio slightly increases. The lower growth rate also implies that in the future operating income is lower. Therefore, shareholders default earlier and the firm value decreases with the payout rate. The principal $\rho(m^*)$ decreases because both the firm value decreases and the maturity increases thereby increasing the default risk on the debt issue. The figure shows that as δ grows and the value of the option to relever decreases the incentive to issue finite maturity debt decreases. Therefore, the firm value and optimal capital structure converges towards the Leland (1994) model.

Increasing the volatility σ of the operating income lowers the firm value because it increases the probability of default. To prevent this default from happening the firm lowers its leverage ratio and increases its debt maturity. Increasing the debt maturity lowers rollover risk, the probability of default at maturity. The higher default probability and increase in debt maturity lead to a decrease in $\rho(m^*)$. The negative relation between volatility and leverage is consistent with many of the articles surveyed in Harris and Raviv (1991) and Parsons and Titman (2008).

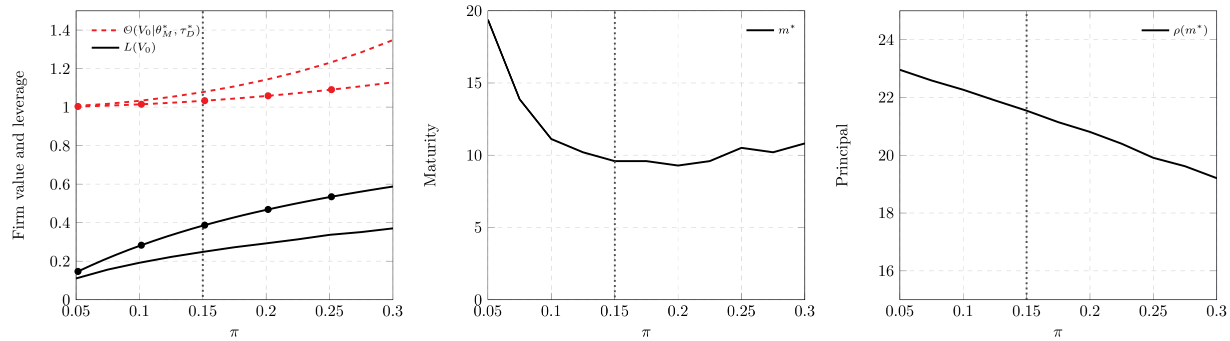
Higher default costs α lead to a lower leverage ratio to prevent default. This lower leverage ratio leads to a lower probability of default. For small default costs, the debt maturity increases to decrease rollover risk and therefore the probability of default. For larger default costs, the effect of the decrease in leverage is so strong that shareholders have less need to worry about rollover risk, which leads to a decrease in the debt maturity. The lower leverage and shorter debt maturity lead to an increase in the principal. If α is an inverse measure of asset tangibility then the negative relation between default costs and leverage is consistent with Frank and Goyal (2009), Rajan and Zingales (1995), Harris and Raviv (1991), and Titman and Wessels (1988).

Issuance costs q are spread out over the life of the debt. This implies that higher issuance costs increase the debt maturity because it takes more time to recuperate the issuance costs. The higher debt maturity leads to a slight increase in leverage because of the deferred possibility to alter the debt. The principal decreases because a longer maturity implies more default risk on a single debt issue.

Finally, the tax benefits of debt π positively influence the firm value. In order to profit from the higher tax benefits, the firm increases its leverage. The higher tax benefits also imply that it takes less time to earn back the debt issuance costs and therefore the debt maturity decreases. The opposite effect of the issuance costs q. For larger tax benefits, and therefore higher leverage, rollover risk also starts to play a role and shareholders start increasing the debt maturity again. The higher leverage ratio leads to a higher default probability and therefore a lower principal. The negative effect of tax benefits on debt maturity is something Stohs and Mauer (1996) find. The positive relation between tax benefits and leverage is in accordance with the empirical evidence in Graham (1996a), Graham et al. (1998), and



(a) Issuance costs q



(b) Tax benefits π

FIGURE 5: The effect of changes in the issuance costs q and the tax benefits π on the optimal firm value $\mathbb{O}(V_0|\theta_M^*, \tau_D^*)$, the optimal leverage $L(V_0)$, the optimal debt maturity m^*, and the principal per unit of coupon $\rho(m^*)$. The marked lines depict the comparative static results for the Leland (1994) model.

Doidge and Dyck (2013).

III. Capital Supply Uncertainty

Fluctuations in capital supply influence capital structure decisions, see Faulkender and Petersen (2006), Leary (2009), and Kisgen (2006). One source of fluctuations in capital supply, examined by Greenwood et al. (2010), Badoer and James (2015), and Graham et al. (2014), are changes in government debt issuance. When the government issues more long-term debt, corporations issue more short-term debt, and when governments issue more debt, corporations issue less debt. There is a crowding-out effect, the government funding needs decrease or increase the supply of capital available to firms in certain parts of the capital market. This section incorporates time-varying capital supply into the model through the cost of capital channel and studies its effects on optimal capital structure.

Time-varying capital supply is modelled using a continuous time two-state Markov chain.

This approach is similar to the way Hackbarth et al. (2006), Bolton et al. (2013), Bhamra et al. (2010), Chen et al. (2013), and Chen et al. (2014) incorporate macroeconomic states of the economy. There is a high state h in which the debt issuance costs are high q_h because of a low supply of capital. There is also a low state in which the debt issuance costs $q_l < q_h$ are low.[5] The transition intensities between the high and low state are given by κ_j with $j \in \{h, l\}$. Given that we are in the low state, the probability that the economy moves in the next instant dt to the high state is $\kappa_l dt$. These intensities imply that in the steady state on average a firm spends a fraction,

$$\frac{\kappa_l}{\kappa_l + \kappa_h},$$

of the time in the high state and the rest of the time in the low state. It also implies that when the firm is in the high (low) state the expected time until it moves to the low (high) state is given by $1/\kappa_h$ $(1/\kappa_l)$.

Time-varying debt issuance costs $q_t \in \{q_h, q_l\}$ lead to a time-varying optimal capital structure choice by shareholders. As opposed to Chen et al. (2013), the firm does not alter its debt maturity at the jumps of the Markov chain but only when the debt matures. Furthermore, shareholders do not commit to an issuance strategy. At each maturity date, shareholders choose their optimal capital structure taking into account the possibility of changes in the debt issuance costs.

The debt issuance costs q_t is an extra state variable in this model besides V_t, C_t, and m_t. After the principal is repaid, the optimal issuance strategy depends on V_t and q_t. As before we are a looking for a homogeneous and Markovian issuance strategy that in this case also depends on q_t. This implies that there will be two optimal issuance strategies $\theta_M^h = (Z_M^h, m_M^h)$ and $\theta_M^l = (Z_M^l, m_M^l)$, with $\theta_M = (\theta_M^h, \theta_M^l)$. As before, in both states it is assumed that debt is issued at par $\rho_j(m)$, with $j \in \{h, l\}$, for each maturity m, see equation (5).

To give more intuition for this issuance strategy, assume we are in the low state at time 0 and θ_M^l debt is issued. This issuance strategy implies that the debt has a coupon $C_0 = Z^{\theta_M^l} V_0$ and maturity $m^{\theta_M^l}$. From that moment onwards three things can happen:

1. The firm defaults τ_D, with the cash flows as before.

2. The debt matures τ_{mat}^1. The firm repays the principal $\rho_l(m_{\tau_{mat}^{1-}})C_{\tau_{mat}^{1-}}$ and issues debt given the issuance strategy θ_M^l.

3. The debt issuance costs change τ_κ^1 to the high state q_h. In this last case there are again three possibilities: the firm defaults τ_D, the debt matures τ_{mat}^1, or the state changes τ_κ^2

[5] This extension of the model, as the model with constant issuance costs, also allows for q_t to depend on m. In that case the issuance costs also depened on the maturity of the debt issued.

towards the low state q_l. If the *firm defaults* τ_D then the cash flows are as before. If the *debt matures* τ^1_{mat} then the firm repays the principal $\rho_l(m_{\tau^{1-}_{mat}})C_{\tau^{1-}_{mat}}$ and issues debt given the issuance strategy θ^h_M. The principal $\rho_l(m_{\tau^{1-}_{mat}})$ is used because the debt issued at time zero had this principal. Because the default decision at maturity depends on the principal, the default strategy depends on the state in which the debt was issued. If the *debt issuance costs change* for the second time τ^2_κ, we are again back at the three possibilities enumerated above.

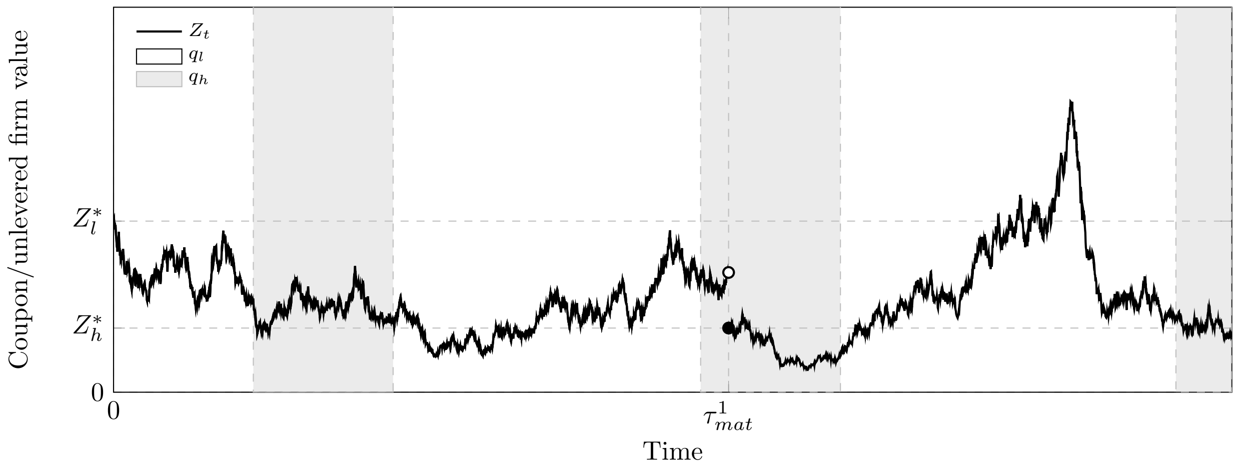
Fig. 6 plots two evolutions of $Z = C/V$ and q over time and depicts the above discussed cases with equilibrium issuance strategies $\theta^*_h = (Z^*_h, m^*_h)$ and $\theta^*_l = (Z^*_l, m^*_l)$. In both figures, the firm is initial in the low state and issues debt with coupon $Z^*_l V_0$ and maturity m^*_l. In Fig. 6(a) the debt matures in the high state and therefore shareholders issue debt with coupon $Z^*_h V_{\tau^1_{mat}}$, lower than $Z^*_l V_{\tau^1_{mat}}$, and maturity m^*_h and in Fig. 6(b) the debt matures in the low state and shareholders issue debt with the same coupon over unlevered firm value and maturity as at time 0. Furthermore, in Fig. 6(a) the debt is restructured downward while in Fig. 6(b) shareholders default because the operating income δV gets too low relative to the coupon payment C, or equivalently Z gets too large.

The default strategy in this model combines the state dependent default barriers of Hackbarth et al. (2006), Bhamra et al. (2010), Chen et al. (2013), and Chen et al. (2014) with the default strategy of Theorem 1. As mentioned before this default strategy depends on the current state given by Z_t and q_t but also on the state when the debt was issued, call this $\hat{q}$. This implies that there are four different default strategies, all similar to the default strategy of the original model, given in equation (4), which follows from $(q_t, \hat{q}) \in \{q_l, q_h\} \times \{q_l, q_h\}$,

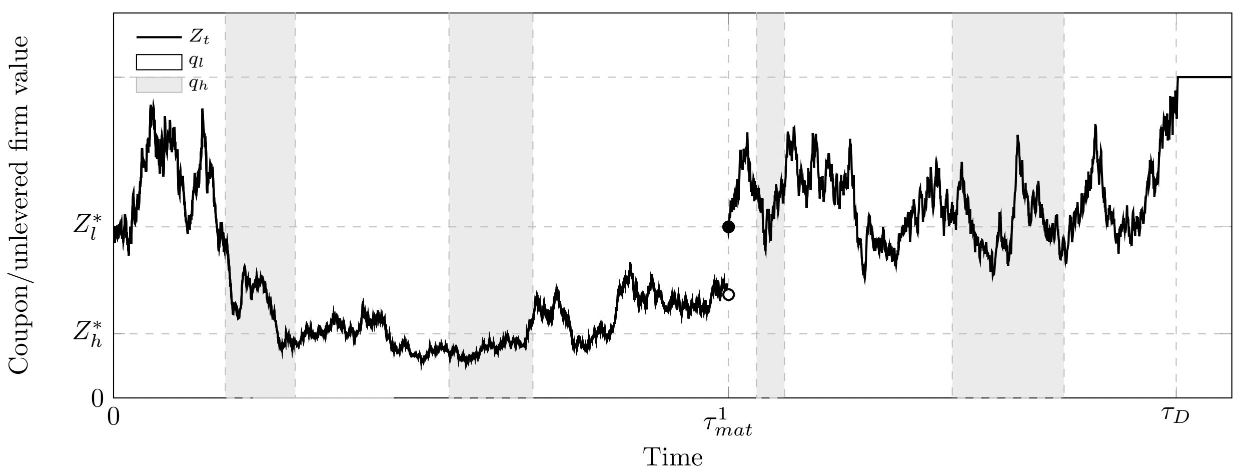
$$\left\{\{q_h, \hat{q}_h\}, \{q_h, \hat{q}_l\}, \{q_l, \hat{q}_h\}, \{q_l, \hat{q}_l\}\right\}.$$

For each of the four possible cases there is an optimal default barrier Z_D and debt maturity default threshold. This gives the solution to the model.

The numerical solution to the model is found by fixing the debt maturity choice of the firm in both states $m^{\theta^j_M}$ and then picking initial values for the optimal firm value $\mathcal{O}(V, q_j|\theta_M, \tau_D)/V = o(q_j|\theta_M, \tau_D)$ and principal $\rho_j(m^{\theta^j_M})$ in both states. Given the initial values, the optimal firm value (and accompanying coupon) in the high state can be calculated, after which the optimal firm value and principal for the high state are updated. The new principal in the high state is the debt value divided by the coupon at the firm value maximizing coupon. This process is repeated for the low state. The two steps are iterated over until the solution converges. The model is solved for a two-dimensional grid of debt maturities. The grid is two-dimensional because shareholders have to make a debt maturity



(a) High state maturity



(b) Low state maturity

FIGURE 6: Two evolutions of Z_t and q_t. The state can alter between the white fill low state q_l and the grey fill high state q_h. Depending on the state at the moment the debt matures the firm changes its issuance strategy. In subfigure b, at the end the operating income gets too small relative to the coupon, Z_t gets too large, and shareholders decide to default.

choice when issuing debt in both the high and low state.

The model is solved for the base case parameters of Table 1 and the time-varying issuance costs parameters of Table 3. The issuance costs in the high state are 1.5% and in the low state 0.5%. The switching intensity in the high state is 0.3. The switching intensity in the low state is set such that the expected issuance costs in the steady state is equal to the issuance costs in the base case,

$$0.75 = \frac{\kappa_l q_h + \kappa_h q_l}{\kappa_l + \kappa_h} = \frac{\kappa_l * 1.5 + 0.3 * 0.5}{\kappa_l + 0.3},$$

which implies that the intensity of switching in the low state is $\kappa_l = 0.1$. In the steady state on average 25% of the time the debt issuance costs are 1.5% and 75% of the time 0.5%. Furthermore, on average a firm spends $3.33 = 1/\kappa_h$ years in the high state and $10 = 1/\kappa_l$ years in the low state before switching to the other state.

Parameter	Symbol	Value
High state issuance costs	q_h	0.015
Low state issuance costs	q_l	0.005
High state switching intensity	κ_h	0.3
Low state switching intensity	κ_l	0.1

TABLE 3: The time-varying issuance costs parameters for the model with time-varying capital supply.

Table 4 gives the solution to the model. It is clear from the table that shareholders of the firm respond to the time-varying debt issuance costs. In the state with high issuance costs firm decides to issue less debt with a longer maturity compared to the state with the low issuance costs. The reason is that issuing debt is less profitable. The lower profitability of debt causes shareholders to issues less debt and to lengthen the maturity to have more time to recuperate the issuance costs. Furthermore, the differences are economically significant, the debt maturity goes from 9.6 to 9 years and the leverage increases by approximately 15% for the optimal capital structure when moving from the high to low state. The difference in firm value depending on whether the firm issues debt given high or low issuance costs is 0.35%. If larger capital supply means lower issuance costs for firms then the result on leverage is in accordance with findings by Faulkender and Petersen (2006) and Leary (2009).

When comparing the results with the comparative statics results on the issuance costs q, see Fig. 5(a), it becomes clear that the debt maturity averages out. This happens because shareholders take into account the possibility of changes in the issuance costs over time. In Fig. 5(a) for $q = 0.5\%$ the maturity is around 5 years while for $q = 1.5\%$ the maturity is around 17 years. Furthermore, observe that in the comparative statics results of Fig. 5(a) the leverage ratio is slightly increasing in q but for our model with time-varying capital supply there is a decrease in leverage going from the low to the high state. The intuition for this result follows from the positive relation between debt maturity and leverage found in Fig. 3. Because the debt maturity in the high state is shorter relative to Fig. 5(a) there is less need to increase leverage preemptively.

Finally, it is important to notice that there is no commitment to a debt issuance strategy in my model. Each time shareholders issue new debt they pick the coupon and maturity that maximizes their claim at that moment in time. In many of the existing dynamic models that look at firms' leverage and maturity decisions shareholders ex ante commit to a fixed debt maturity or issuance strategy. In practice shareholders do not commit to a debt maturity or issuance strategy ex ante. This property of my model allows me to study time-variation in the firm's optimal debt maturity.

Parameter	Symbol	Value
High state q_h		
Optimal firm value	$\mathcal{O}(V_0, q_h \mid \theta_M^*, \tau_D^*)$	1.076
Leverage	$L(V_0)$	0.220
Debt maturity	m_h^*	9.63
Coupon	Z_h^*	0.0108
Low state q_l		
Optimal firm value	$\mathcal{O}(V_0, q_l \mid \theta_M^*, \tau_D^*)$	1.079
Leverage	$L(V_0)$	0.255
Debt maturity	m_l^*	8.966
Coupon	Z_h^*	0.0128

TABLE 4: The model implied parameters for the model with time-varying capital supply.

IV. Conclusion

This article develops a dynamic capital structure model in which shareholders determine a firm's leverage ratio, debt maturity, and default strategy. The firm has a single debt issue outstanding and therefore all debt matures at once. After repaying the principal, shareholders own all the firm's cash flows and can alter the firm's capital structure. The possibility to alter the capital structure at maturity gives shareholders the incentive to issue finite maturity debt and allows me to study firms' joint choice of leverage and debt maturity. Based on this model, I then produce a new set of testable implications and replicate empirical patterns on firms' choice of leverage and debt maturity. Furthermore, the model shows that leverage and debt maturity interact and that to understand firms' capital structure decisions it is of first-order importance to look at the joint choice of leverage and debt maturity. Finally, I extend the model by making capital supply time-varying, which allows me to study time-variation in firms' joint choice of leverage and debt maturity.

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Appendix

The first section of the appendix contains the results for the extension of the model where shareholders are allowed to restructure their debt upon meeting new creditors as in Hugonnier et al. (2015). The second section contains the proofs of all the articles results.

A. Debt Restructuring

This section extends the model to allow shareholders to restructure their debt upward in the presence of capital supply uncertainty. Shareholders can change their leverage ratio not only at maturity dates but also upon meeting new creditors. This setup is similar to Hugonnier et al. (2015) and the seminal work by Fischer et al. (1989). As for the original model, first the model is described after which the game between shareholders and creditors is formulated and a solution is given. Finally, the model is analyzed.

Contrary to Fischer et al. (1989), in my setup shareholders can only restructure their debt upwards. Gilson (1997) shows that in practice transaction costs discourage shareholders from reducing leverage. These transaction costs partially follow from the holdout problem with existing creditors, see Hugonnier et al. (2015). At maturity this holdout problem no longer exists because the debt value is the principal and therefore at maturity shareholder can restructure debt both upward and downward. This gives shareholders an incentive to issue finite maturity debt. Creditors are often dispersed, finding them and buying back their debt is costly and adds to the restructuring costs. Therefore, when shareholders restructure their debt, they incur restructuring costs ϵV.

To incorporate capital supply uncertainty shareholders can only restructure their debt when they meet new creditors. These meetings happen at the jumps of a Poisson process with intensity λ, so on average every $1/\lambda$ period shareholders meet new creditors. The parameter λ is the intensity with which shareholders meet new creditors. The random time τ_λ^i is the ith time that shareholders meet new creditors. Hugonnier et al. (2015) have the same setup to model capital supply uncertainty. The coupon can change both at maturity dates τ_{mat}^i and upon meeting new creditors τ_λ^i instead of only at maturity dates as in the original model. In the limit when λ goes to infinity shareholders continuously meet new creditors and can at any instant restructure their debt.

As in Fischer et al. (1989), Goldstein et al. (2001), Strebulaev (2007), and Hugonnier et al. (2015), I assume has to be bought back at market value. The debt claim of existing creditors is valued assuming all surplus at the moment of restructuring goes to shareholders. Therefore, the debt value is as in the original model (1) where debt is price competitively and is only indirectly influenced by the option to restructure, through changes in the default time τ_D. Let a be an $\mathcal{F}$-adapted upward restructuring strategy.

To simplify notation, the firm value at the moment of debt issuance (net of issuance costs) is,

$$F(V_t, C_{t^-}, m_t|\theta, a, \tau_D) = E(V_t, C_{t^-}, m_t|\theta, a, \tau_D) + (1-q)D(V_t, C_{t^-}, m_t|\tau_D).$$

The surplus of restructuring at the moment shareholders meet new creditors is,

$$\begin{aligned}\mathcal{S}(V_t, C_{t^-}, m_t|\theta, a, \tau_D) = \sup_{a_{t^-}\geq 1} \Big\{ & F(V_t, a_{t^-}C_{t^-}, m_t|\theta, a, \tau_D) \\ & - \Big(F(V_t, C_{t^-}, m_t|\theta, a, \tau_D) + \mathbb{1}_{\{a_{t^-}>1\}} qD(V_t, C_{t^-}, m_t|\tau_D)\Big) \\ & - \mathbb{1}_{\{a_{t^-}>1\}} \epsilon V_t \Big\}. \end{aligned}$$

If shareholders increase the leverage ratio $a_{t^-} > 1$, they lose their current claim $E(V_t, C_{t^-}, m_t|\theta, a, \tau_D)$ and have to buy their debt back at the market price $D(V_t, C_{t^-}, m_t|\tau_D)$ (second term). Furthermore, they have to pay restructuring costs ϵV_t (third term). In return shareholders get the new equity claim plus the debt proceeds $F(V_t, a_{t^-}C_{t^-}, m_t|\theta, a, \tau_D)$ (first term). Observe that when debt is restructured shareholders can only issue debt with the same maturity.[6] If $a_{t^-} = 1$ then the surplus is zero. Shareholders only restructure their debt when they meet new creditors and the surplus is strictly positive. The equity value at the moment of restructuring is the surplus plus the equity value,

$$\mathcal{R}(V_t, C_{t^-}, m_t|\theta, a, \tau_D) = \mathcal{S}(V_t, C_{t^-}, m|\theta, a, \tau_D) + E(V_t, C_{t^-}, m_t|\theta, a, \tau_D).$$

Fig. 7 explains how debt restructuring works. The left side contains the original model and is exactly the same as before, see Fig. 2. The right side describes the restructuring process. Assume the firm has debt outstanding and is paying dividends and coupon. From then onwards three things can happen, either the firm defaults, debt matures, or shareholders meet new creditors. The first two cases are as in the original model. In the third case, when shareholders meet new creditors, they decide whether to restructure their debt. This choice depends on whether the surplus of restructuring $\mathcal{S}(V_{\tau_\lambda^i}, C_{\tau_\lambda^{i-}}, m_{\tau_\lambda^i}|\theta, a, \tau_D)$ is positive. Let $a_{\tau_\lambda^{i-}}$ be the solution to this optimization problem. At this moment shareholders claim is worth $\mathcal{R}(V_{\tau_\lambda^i}, C_{\tau_\lambda^{i-}}, m_{\tau_\lambda^i}|\theta, a, \tau_D)$, which is the equity value plus the restructuring surplus and creditors claim is $D(V_{\tau_\lambda^i}, C_{\tau_\lambda^{i-}}, m_{\tau_\lambda^i}|\tau_D)$. If the surplus is zero shareholders part with the new creditors and the firm continues to pay dividends and coupon. If the surplus is strictly positive then shareholders buy back the debt at the market price $D(V_{\tau_\lambda^i}, C_{\tau_\lambda^{i-}}, m_{\tau_\lambda^i}|\tau_D)$ and pay the restructuring costs $\epsilon V_{\tau_\lambda^i}$. After the debt is bought back, there is no more debt outstanding and shareholders own the entire firm. In this case shareholders claim is $F(V_{\tau_\lambda^i}, a_{\tau_\lambda{}^{i-}} C_{\tau_\lambda^{i-}}, m_{\tau_\lambda^i}|\theta, a, \tau_D)$ where $a_{\tau_\lambda^{i-}} C_{\tau_\lambda^{i-}}$ is the optimal coupon and creditors claim is zero. Shareholders issue debt with this larger coupon and obtain the proceeds $(1-q)D(V_{\tau_\lambda^i}, a_{\tau_\lambda^{i-}} C_{\tau_\lambda^{i-}}, m_{\tau_\lambda^i}|\tau_D)$ while credi-

[6] This restriction is necessary for tractability, else the debt maturity might change over time and depend on the moment shareholders meet new creditors.

tors buy the debt $D(V_{\tau_\lambda^i}, a_{\tau_\lambda^{i-}} C_{\tau_\lambda^{i-}}, m|\tau_D)$ and then the firm returns to paying dividends and coupon.

This implies that the equity value is defined as,

$$
\begin{aligned}
E(V_0, C_0, m_0|\theta, a, \tau_D) =& \mathbb{E}_0\left[\int_0^{\tau_D \wedge \tau_{mat}^1 \wedge \tau_\lambda^1} e^{-rt}\left(\delta V_t - (1-\pi)C_t\right) dt\right] \\
&+ \mathbb{E}_0\left[e^{-r\tau_{mat}^1} \mathbb{1}_{\left\{\tau_{mat}^1 < \tau_D \wedge \tau_\lambda^1\right\}}\left(\mathcal{O}(V_{\tau_{mat}^1}|\theta, a, \tau_D) - \rho C_{\tau_{mat}^{1-}}\right)\right] \\
&+ \mathbb{E}_0\left[e^{-r\tau_\lambda^1} \mathbb{1}_{\left\{\tau_\lambda^1 < \tau_D \wedge \tau_{mat}^1\right\}} \mathcal{R}\left(V_{\tau_\lambda^1}, C_{\tau_\lambda^{1-}}, m_{\tau_\lambda^1}|\theta, a, \tau_D\right)\right].
\end{aligned}
$$

Shareholders receive dividends until either the debt matures, the firm defaults, or they meet new creditors (first term). If the firm defaults shareholders get nothing and in case the debt matures they get the optimal firm value minus the principal $\mathcal{O}(V_{\tau_{mat}^1}|\theta, a, \tau_D) - \rho C_{\tau_{mat}^{1-}}$ (second term). If shareholders meet new creditors they get the equity value at restructuring $\mathcal{R}(V_{\tau_\lambda^1}, C_{\tau_\lambda^{1-}}, m_{\tau_\lambda^1}|\theta, a, \tau_D)$ (third term).

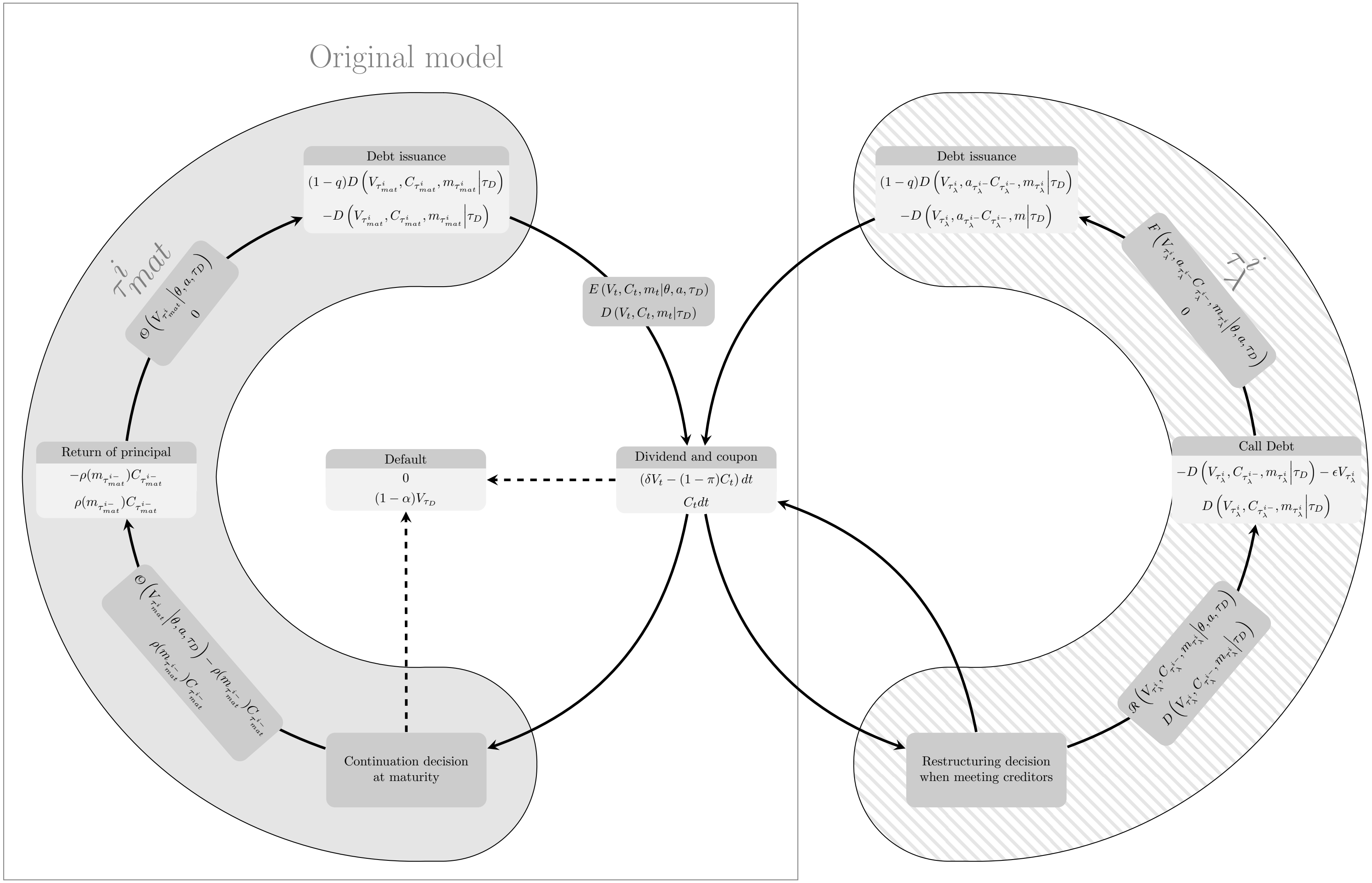


FIGURE 7: The cash flows (the two toned boxes) to shareholders and creditors and the value of shareholders and creditors claim (single toned boxes). The top equation is for shareholders while the bottom equation is for creditors. The gray shaded area depicts the maturity date τ^i_m and the gray striped area the restructuring date τ^i_λ.

A. Shareholders vs Creditors

As in the original model creditors have to conjecture shareholders' default strategy to value the debt claim. The game is an infinitely repeated game with discounting that stops at the moment shareholders default. This section look for a homogeneous Markov perfect equilibrium. The constituent game is from the moment the firm issues debt until it repays the principal, restructures its debt, or defaults. The only difference with the original model is that shareholders can increase leverage when they meet new creditors:

1. Shareholders issue debt with coupon $Z_0^\theta V_0$ and maturity m_0^θ.
2. Competetive creditors buy the debt for a price $P(V_0, Z_0^\theta V_0, m_0^\theta)$ and set $i = 1$.
3. Shareholders determine a default strategy and the firm pays dividends and coupon until $\tau_{mat}^1 \vee \tau_\lambda^i \vee \tau_D$.
4. If $\tau_{mat}^1 < \tau_\lambda^i \vee \tau_D$ shareholders return the principal $\rho C_{\tau_{mat}^{1-}}$ if profitable and go to step 1. If $\tau_\lambda^i < \tau_{mat}^1 \vee \tau_D$ and the restructuring surplus is zero then $i = i+1$ and shareholders go to step 3. If the surplus is positive then shareholders call the debt for $P(V_{\tau_\lambda^i}, C_{\tau_\lambda^{i-}}, m_{\tau_\lambda^i})$, pay the restructuring costs $\epsilon V_{\tau_\lambda^i}$, and go to step 1. In step 1 the debt maturity is fixed at m_0^θ and shareholders need to increase the coupon.[7] If $\tau_D < \tau_{mat}^1 \vee \tau_\lambda^i$ then shareholders default, the bankruptcy proceeds $(1-\alpha)V_{\tau_D}$ go to creditors and the game ends.

Shareholders strategy in one constituent game exists of picking a coupon Z_0^θ, debt maturity m_0^θ, restructuring strategy a, and default strategy τ_D. Creditors strategy is setting a price $P(\cdot,\cdot,\cdot)$ for any debt issuance strategy (Z_0^θ, m_0^θ) given V_0.

Definition 3. *A Markov perfect equilibrium is a debt issuance strategy θ_M, restructuring strategy a, default strategy τ_D, debt price $P(\cdot,\cdot,\cdot)$ that are Markovian such that:*

1. *Shareholders maximize the equity value plus the net proceeds at maturity dates, the restructuring surplus upon meeting new creditors, and the equity value at any other date.*
2. *Creditors make zero profit in expectation on each debt issue.*

As in the original model the equilibrium debt price is given by equation (1). This price ensures that creditors are competitive and make zero profit in equilibrium.

[7] I would lose tractability if I allow shareholders to also alter the debt maturity at restructuring dates.

B. Optimal Restructuring Strategy

The optimal restructuring strategy is first examined before the existence of a Markov perfect equilibrium is shown.

Proposition 3. *For any (θ_M, τ_D) such that τ_D is given by equation (4), one of the following two restructuring strategies is optimal:*

1. ***Barrier strategy**: $a > 0$ when $Z \in [0, Z_r^{bar})$ and else $a = 1$.*
2. ***Band-barrier strategy**: $a > 0$ when $Z \in [0, Z_r^{bar}) \cup (Z_r^{bnd-}, Z_r^{bnd+})$ and else $a = 1$.*

Where Z^{bar}, Z_r^{bnd-}, and Z_r^{bnd+}, with $Z^{bar} < Z_r^{bnd-} < Z_r^{bnd+}$, are thresholds at which shareholders are indifferent between restructuring and not altering the firm's capital structure. For large enough restructuring costs ϵ, $Z_r^{bar} = 0$ and shareholders never restructure the firm's debt.

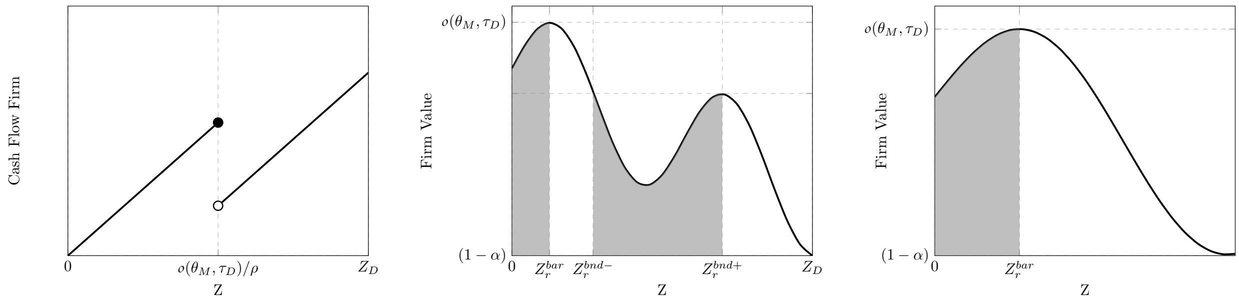
The barrier restructuring strategy says that when the ratio of coupon over unlevered firm value Z is below the barrier Z_r^{bar} and shareholders meet new creditors then the firm restructures its debt. The band-barrier restructuring strategy says that when the ratio of coupon over unlevered firm value Z is either below the barrier Z_r^{bar} or in the band (Z_r^{bnd-}, Z_r^{bnd+}) and shareholders meet new creditors then the firm restructures its debt.

The existence of two optimal restructuring strategies follows from the fact that the cash flow of the firm (without restructuring) with the debt maturity integrated out,

$$\begin{aligned}
&\delta V_t + \pi C_t + \eta \left(\mathcal{O}(V_t|\theta_M, a, \tau_D) + \mathbb{1}_{\{\mathcal{O}(V_t|\theta_M,a,\tau_D)-\rho C_t\}} \left((1-\alpha)V_t - \mathcal{O}(V_t|\theta_M, a, \tau_D)\right)\right) \\
&= V_t \left\{ \delta + \pi Z_t + \eta \left(o(\theta_M, a, \tau_D) + \mathbb{1}_{\{o(\theta_M,a,\tau_D)-\rho Z_t\}} \left((1-\alpha) - o(\theta_M, a, \tau_D)\right)\right)\right\}. \qquad (6)
\end{aligned}$$

is discontinues in Z_t at the point where shareholders are indifferent between defaulting at maturity and continuing to operate at maturity, as in the left figure of Fig. 8. This discontinuity can lead to a double hump shaped firm value, see the middle figure, and optimality of the band-barrier strategy. For low leverage ratios, low values of Z it is always optimal to restructure the debt towards the optimal leverage ratio and this leads to the restructuring barrier Z_r^{bar}.

The longer the debt maturity the less important the discontinuity at $o(\theta_M, a, \tau_D)/\rho$ becomes, see equation (6) for increasing $m = 1/\eta$. This effect ensures that firm value becomes single hump shaped. Economically it means that the longer the debt maturity the less important the payoff and possibility of default at maturity becomes. It causes the barrier strategy to be optimal. For longer maturities the cash flow of the firm also converges towards the



(a) Cash Flow (b) Band-barrier strategy (c) Barrier strategy

FIGURE 8: The cash flow to the firm (left) and the two possible optimal restructuring strategies given two possible firm values (middle and right) assuming there are no issuance $q = 0$ and restructuring costs $\epsilon = 0$.

cash flow of the firm in Hugonnier et al. (2015), for which the optimal restructuring strategy is the barrier strategy. The following proposition makes this rigorous,

Proposition 4. *For issuance strategies with a sufficiently long debt maturity the optimal restructuring strategy is a barrier strategy.*

In the numerical analysis done below shareholders always use a barrier restructuring strategy and it is never optimal to use a band-barrier restructuring strategy.

C. Model Solution

Given the optimal restructuring strategy, the final step shows that a finite optimal firm value and default threshold exist. To prove Theorem 1 for the case with restructuring the following assumption is necessary,

Assumption 1. *For any $\theta_M \in \mathbb{R}^2_+$ and one-shot first period deviation of θ_M, the equity value is non-negative given the default boundary that satisfies the smooth pasting condition. Furthermore, the equity value is non-decreasing in the firm value the next iteration for any Markovian strategy θ_M.*

Theorem 2 below is the equivalent of Theorem 1 when allowing for debt restructuring,

Theorem 2. *For $\pi < \pi^*(\pi)$ such that Assumption 1 holds there exists a Markov Perfect Equilibrium. In this equilibrium,*

- *The equilibrium issuance strategy θ_M^* maximizes the firm value assuming shareholders default optimally.*

- *The equilibrium default strategy* τ_D^* *is,*

$$\tau_D^* = \inf \{\mathcal{C}, \mathcal{P}\},$$

where the default barrier Z_D^* *satisfies the smooth pasting condition.*

- *The equilibrium debt price equals the expected discounted cash flows of the debt given* τ_D^*.

- *The equilibrium restructuring strategy* a^* *is given by Proposition 3.*

- *Off-equilibrium, creditors belief that any deviation from the optimal issuance strategy is a one-shot deviation. The off-equilibrium debt price is the expected discounted cash flows of the debt given the optimal default strategy that follows from these beliefs.*

D. Model Analysis

Table 1 and Table 5 give the base case parameters for the model analysis. The meeting intensity is set equal to 1, which implies that on average every year shareholders meet new creditors and have the possibility to restructure their debt. The restructuring costs are set equal to 1.125% of asset value. As before debt is issued at par for each maturity given the optimal coupon, see equation (5).

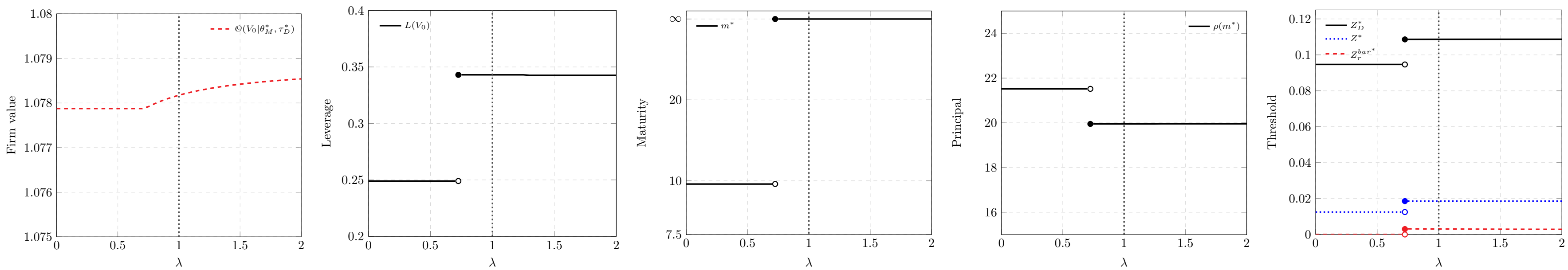
Parameter	Symbol	Value
Meeting intensity	λ	1
Restructuring costs	ϵ	0.01125

TABLE 5: The restructuring parameters for the model with restructuring.

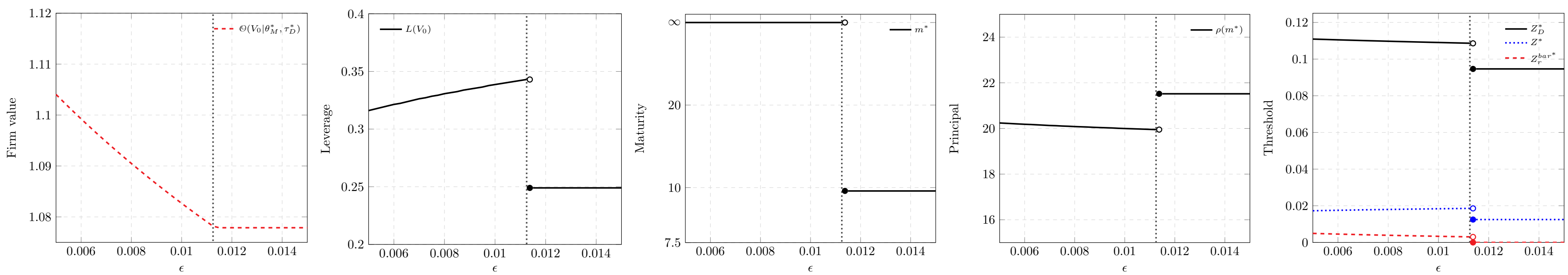
Fig. 9 shows a comparative statics analysis with respect to the meeting intensity λ and the restructuring costs ϵ. For the base case parameters the firm is approximately indifferent between using the option to relever or restructuring at maturity. From Fig. 9(a) it becomes clear that increasing the meeting intensity increases the firm value but at a decreasing marginal rate when the firm restructures. The reason is that the marginal gain of more often meeting new creditors decreases. Decreasing the expected interval between meeting new creditors $\mathbb{E}_{\tau_\lambda^i}[\tau_\lambda^{i+1} - \tau_\lambda^i] = 1/\lambda$ decreases the expected change and volatility of the firm value in this interval. This decrease in expected change and volatility decreases the option value of restructuring. Furthermore, the firm issues perpetual debt when it starts using the option to restructure and therefore no longer uses the option to relever at maturity.

The comparative statics results for the restructuring costs have a similar intuition. For low restructuring costs, increasing the restructuring costs lowers the firm value. As the costs become larger and larger the chances of shareholders actually restructuring the firm's debt decrease. Therefore, the marginal change in firm value of increasing the restructuring costs decrease as well. For high restructuring costs shareholders abstain from restructuring and only rely on the option to relever at maturity. In this region the optimal debt maturity becomes finite again.

The comparative statics analysis shows that for sufficient restructuring costs shareholders still have the incentive to issue finite maturity debt. Furthermore, the higher the restructuring costs the lower the incentives for shareholders to restructure their debt and the more they prefer to wait until the debt matures. Finally, increasing the chances of meeting new creditors is beneficial for shareholders but at a marginally decreasing rate.



(a) Meeting intensity λ



(b) Restructuring costs ϵ

FIGURE 9: The effects of changes in the meeting intensity λ and the restructuring costs ϵ on the optimal firm value $\mathcal{O}(X_0|\theta_M^*, \alpha^*, \tau_D^*)$, the optimal leverage $L(V_0)$, the optimal debt maturity m^*, the principal $\rho(m^*)$, and thresholds $(Z_D^*, Z^*, Z_r^{bar^*})$.

B. Model Solution

The proofs are organized as follows:

- The model (with restructuring) is homogeneous in V and C. Under a Markovian default strategy τ_D the equity and debt value can be rewritten as functions of $Z = C/V$.

- For a given m, the firm value $f(Z, m|\theta_M, a, \tau_D) = e(Z, m|\theta_M, a, \tau_D) + (1-q)d(Z, m|\tau_D)$ is a contraction mapping for the default strategy τ_D. Where τ_D is given by a barrier G at maturity, such that shareholders default when the principal exceeds G. At all other times shareholders default when Z exceeds Z_D. The optimal default barrier at maturity G is given by the optimal firm value and is fixed at this value.

- The optimal restructuring strategy is characterized. It is either a barrier strategy implying that shareholders restructuring when $Z \in [0, Z^r_{bar})$ and they meet a creditor or a barrier-band strategy where shareholders restructure when $Z \in [0, Z^r_{bar}) \cup (Z^r_{bnd-}, Z^r_{bnd+})$ and they meet a creditor. Furthermore, for long debt maturities the barrier strategy is the optimal restructuring strategy.

- For any m, the optimal firm value is finite for tax benefits $\pi < \pi^*(\pi)$. Because the firm value is finite given any debt maturity m there exists a default threshold that satisfies the smooth pasting condition. Furthermore, this default threshold leads to the optimal default strategy.

- The one-shot deviation principle shows existence of the Markov Perfect Equilibrium of Theorem 1 and Theorem 2.

- For small principals $\rho(m)$ ans issuance cost q the optimal debt maturity is finite in the equilibrium of Theorem 1.

It is important to remember that $m = 1/\eta$, so η is the intensity with which the debt matures.

A. Homogeneity

The first step is to show that the debt and equity value are homogeneous of degree one in V and C,

Lemma 1. *If shareholders debt issuance strategy θ_M and default strategy τ_D are homogeneous and Markovian then the equity and debt value are homogeneous of degree one in V and C*

with,

$$E(V,C,m|\theta_M,a,\tau_D) = Ve(Z,m|\theta_M,a,\tau_D),$$
$$D(V,C,m|\tau_D) = Vd(Z,m|\tau_D).$$

Similarly, the optimal firm value $\mathcal{O}(V|\theta_M,a,\tau_D) = V\mathcal{o}(\theta_M,a,\tau_D)$ *and the firm value at restructuring* $\mathcal{R}(V,C,m|\theta_M,a,\tau_D) = V\mathcal{r}(Z,m|\theta_M,a,\tau_D)$ *are defined as,*

$$\mathcal{o}(\theta_M,a,\tau_D) = e(Z^{\theta_M},m^{\theta_M}|\theta_M,a,\tau_D) + (1-q)d(Z^{\theta_M},m^{\theta_M}|\tau_D),$$
$$\mathcal{r}(Z,m|\theta_M,a,\tau_D) = \sup_{a'\geq 1}\left\{e(a'Z,m|\theta_M,a,\tau_D) + \left(1-q\mathbb{1}_{\{a'>1\}}\right)d(a'Z,m|\tau_D) - d(Z,m|\tau_D) - \mathbb{1}_{\{a'>1\}}\epsilon\right\}.$$

The equity value becomes,

$$\begin{aligned} e(Z_0,m_0|\theta_M,a,\tau_D) =& \tilde{\mathbb{E}}_0\left[\int_0^{\tau_D\vee\tau^1_{mat}\vee\tau^1_\lambda} e^{-\delta t}\left(\delta-(1-\pi)Z_t\right)dt\right] \\ &+ \tilde{\mathbb{E}}_0\left[\mathbb{1}_{\left\{\tau^1_{mat}<\tau_D\vee\tau^1_\lambda\right\}}e^{-\delta\tau^1_{mat}}\left(\mathcal{o}(\theta_M,a,\tau_D)-\rho(m_0)Z_{\tau^{1-}_{mat}}\right)\right] \\ &+ \tilde{\mathbb{E}}_0\left[\mathbb{1}_{\left\{\tau^1_\lambda<\tau_D\vee\tau^1_{mat}\right\}}e^{-\delta\tau^1_\lambda}\mathcal{r}\left(Z_{\tau^{1-}_\lambda},m_0|\theta_M,a,\tau_D\right)\right], \end{aligned}$$

and the debt value,

$$d(Z_0,m_0|\tau_D) = \tilde{\mathbb{E}}_0\left[\int_0^{\tau_D\vee\tau^1_{mat}} e^{-\delta t}Z_t dt + \mathbb{1}_{\left\{\tau^1_{mat}<\tau_D\right\}}e^{-\delta\tau^1_{mat}}\rho(m_0)Z_{\tau^{1-}_{mat}} + \mathbb{1}_{\left\{\tau_D\leq\tau^1_{mat}\right\}}e^{-\delta\tau_D}(1-\alpha)\right],$$

with the dynamics of Z_t*,*

$$dZ_t = -(r-\delta)Z_t dt - \sigma Z_t d\tilde{B}_t.$$

Proof. Rewrite the equity value (3),

$$
\begin{aligned}
E(V_0, C_0, m_0|\theta_M, a, \tau_D) =& \mathbb{E}_0\left[\int_0^{\tau_D \vee \tau^1_{mat} \vee \tau^1_\lambda} e^{-rt}\left(\delta V_t - (1-\pi)C_t\right) dt\right] \\
&+ \mathbb{E}_0\left[\mathbb{1}_{\left\{\tau^1_{mat} < \tau_D \vee \tau^1_\lambda\right\}} e^{-r\tau^1_{mat}}\left(\mathcal{O}\left(V_{\tau^1_{mat}}|\theta_M, a, \tau_D\right) - \rho(m_0)C_{\tau^{1-}_{mat}}\right)\right] \\
&+ \mathbb{E}_0\left[\mathbb{1}_{\left\{\tau^1_\lambda < \tau_D \vee \tau^1_{mat}\right\}} e^{-r\tau^1_\lambda}\mathcal{R}\left(V_{\tau^1_\lambda}, C_{\tau^{1-}_\lambda}, m_0|\theta_M, a, \tau_D\right)\right] \\
=& \mathbb{E}_0\left[\int_0^{\tau_D \vee \tau^1_{mat} \vee \tau^1_\lambda} e^{-rt} V_t\left(\delta - (1-\pi)Z_t\right) dt\right] \\
&+ \mathbb{E}_0\left[\mathbb{1}_{\left\{\tau^1_{mat} < \tau_D \vee \tau^1_\lambda\right\}} e^{-r\tau^1_{mat}} V_{\tau^1_{mat}}\left(o(\theta_M, a, \tau_D) - \rho(m_0)Z_{\tau^{1-}_{mat}}\right)\right] \\
&+ \mathbb{E}_0\left[\mathbb{1}_{\left\{\tau^1_\lambda < \tau_D \vee \tau^1_{mat}\right\}} e^{-r\tau^1_\lambda} V_{\tau^1_\lambda}\mathfrak{r}\left(Z_{\tau^{1-}_\lambda}, m_0|\theta_M, a, \tau_D\right)\right] \\
=& V_0\tilde{\mathbb{E}}_0\left[\int_0^{\tau_D \vee \tau^1_{mat} \vee \tau^1_\lambda} e^{-\delta t}\left(\delta - (1-\pi)Z_t\right) dt\right] \\
&+ V_0\tilde{\mathbb{E}}_0\left[\mathbb{1}_{\left\{\tau^1_{mat} < \tau_D \vee \tau^1_\lambda\right\}} e^{-\delta\tau^1_{mat}}\left(o(\theta_M, a, \tau_D) - \rho(m_0)Z_{\tau^{1-}_{mat}}\right)\right] \\
&+ V_0\tilde{\mathbb{E}}_0\left[\mathbb{1}_{\left\{\tau^1_\lambda < \tau_D \vee \tau^1_{mat}\right\}} e^{-\delta\tau^1_\lambda}\mathfrak{r}\left(Z_{\tau^{1-}_\lambda}, m_0|\theta_M, a, \tau_D\right)\right] \\
=& V_0 e(Z_0, m_0|\theta_M, a, \tau_D).
\end{aligned}
$$

For the second equality, apply the change of measure,

$$
\tilde{\mathbb{Q}}(A) = \mathbb{E}\left[\mathbb{1}_{\{A\}} e^{-(r-\delta)t}\frac{V_t}{V_0}\right] = \mathbb{E}\left[\mathbb{1}_{\{A\}} e^{-\frac{\sigma^2}{2}t + \sigma B_t}\right] \ \forall A \subseteq \mathcal{F}_t. \tag{7}
$$

The dynamics of Z_t follow,

$$
dZ_t = -(r-\delta)dZ_t - \sigma Z_t dB_t + \sigma^2 Z_t dt = -(r-\delta)dZ_t - \sigma Z_t(dB_t - \sigma dt) = -(r-\delta)Z_t dt - \sigma Z_t d\tilde{B}_t.
$$

Rewrite the debt value (1),

$$\begin{aligned}
D(V_0, C_0, m_0|\tau_D) =& \mathbb{E}_0\left[\int_0^{\tau_D \vee \tau^1_{mat}} e^{-rt} C_t dt + \mathbb{1}_{\{\tau^1_{mat} < \tau_D\}} e^{-r\tau^1_{mat}} \rho(m_0) C_{\tau^{1-}_{mat}}\right] \\
&+ \mathbb{E}_0\left[\mathbb{1}_{\{\tau_D \leq \tau^1_{mat}\}} e^{-r\tau_D}(1-\alpha)V_{\tau_D}\right] \\
=& \mathbb{E}_0\left[\int_0^{\tau_D \vee \tau^1_{mat}} e^{-rt} V_t Z_t dt + \mathbb{1}_{\{\tau^1_{mat} < \tau_D\}} e^{-r\tau^1_{mat}} V_{\tau^1_{mat}} \rho(m_0) Z_{\tau^{1-}_{mat}}\right] \\
&+ \mathbb{E}_0\left[\mathbb{1}_{\{\tau_D \leq \tau^1_{mat}\}} e^{-r\tau_D}(1-\alpha)V_{\tau_D}\right] \\
=& V_0 \tilde{\mathbb{E}}_0\left[\int_0^{\tau_D \vee \tau^1_{mat}} e^{-\delta t} Z_t dt + \mathbb{1}_{\{\tau^1_{mat} < \tau_D\}} e^{-\delta\tau^1_{mat}} \rho(m_0) Z_{\tau^{1-}_{mat}}\right] \\
&+ \tilde{\mathbb{E}}_0\left[\mathbb{1}_{\{\tau_D \leq \tau^1_{mat}\}} e^{-\delta\tau_D}(1-\alpha)\right] \\
=& V_0 d(Z_0, m_0|\tau_D)
\end{aligned}$$

The second equality follows from the change of measure (7). □

B. *Existence Firm Value*

The previous lemma shows that given a homogeneous and Markovian issuance strategy θ_M and default strategy τ_D the firm value is homogeneous of degree one in V and C. Let the homogeneous and Markovian default strategy τ_D be given by,

$$\begin{aligned}
\mathcal{C} &= \left\{t > 0 | Z_t \geq Z_D \text{ s.t. } \forall i \in \mathbb{N}\ t \neq \tau^i_{mat}\right\}, \\
\mathcal{P} &= \left\{t > 0 | \rho(m_{t^-}) Z_{t^-} > G, \exists i \in \mathbb{N} \text{ s.t. } t = \tau^i_{mat}\right\}, \\
\tau_D &= \inf\left\{\mathcal{C}, \mathcal{P}\right\}.
\end{aligned}$$

This implies that τ_D is defined by (Z_D, G). From now on assume τ_D is described by (Z_D, G).

Given a default threshold Z_D the firm value is bounded from above by,

$$\frac{\delta + (\pi - q) Z_D}{\delta}, \tag{8}$$

which is the unlevered firm value plus the perpetuity of the maximum tax benefits that can be obtained.

The function $f(Z)$ denotes the firm value for a homogeneous and Markovian issuance strategy θ_M, default strategy τ_D described by (Z_D, G), and restructuring strategy a. Similarly define the abbreviation $e(Z)$ and $d(Z)$ as,

$$
\begin{aligned}
e(Z) &= e(Z, m^{\theta_M} | \theta_M, a, \tau_D), \\
d(Z) &= d(Z, m^{\theta_M} | \theta_M, \tau_D), \\
f(Z) &= e(Z) + (1-q)d(Z).
\end{aligned}
$$

Given the default strategy (Z_D, G) the debt value is,

$$
\begin{aligned}
d(Z) = &\tilde{\mathbb{E}}_0 \left[\int_0^{\tau_D^{\mathcal{C}}} e^{-(\delta+\eta)t} \left(Z_t + \eta \left(\rho(m^{\theta_M}) Z_t \mathbb{1}_{\left\{G \geq \rho(m^{\theta_M}) Z_t\right\}} + (1-\alpha) \mathbb{1}_{\left\{G < \rho(m^{\theta_M}) Z_t\right\}} \right) \right) dt \right] \\
&+ \tilde{\mathbb{E}}_0 \left[e^{-(\delta+\eta)\tau_D^{\mathcal{C}}} (1-\alpha) \right],
\end{aligned}
$$

where $\tau_D^{\mathcal{C}} = \inf\{t > 0 | Z_t \geq Z_D\}$ is the part of τ_D that coincides with the first time hitting the boundary Z_D, see the definition of $\mathcal{C}$ above. The maturity date τ_{mat}^1 is integrated out and creditors value the debt as if restructuring never occurs.

Given a firm value $f(Z)$ the surplus of restructuring is,

$$
\mathfrak{s}(Z) = \sup_{a' \geq 1} \left\{ f(a'Z) - f(Z) - \mathbb{1}_{\{a'>1\}} q d(Z) - \mathbb{1}_{\{a'>1\}} \epsilon \right\}, \tag{9}
$$

in case new debt is issued $(a' > 1)$ the surplus is the new firm value minus issuance cost $f(a'Z)$ minus the current firm value (plus issuance cost) $f(Z) + qd(Z, m | Z_D, G)$ and the restructuring costs ϵ. If no debt is issued $(a' = 1)$ the surplus is zero. Furthermore, the equity value at moment of meeting a creditor is given by the old equity value plus the surplus of restructuring,

$$
\mathfrak{r}(Z) = \mathfrak{s}(Z) + e(Z).
$$

The equity value is defined as,

$$
\begin{aligned}
e(Z_0) = &\tilde{\mathbb{E}}_0 \left[\int_0^{\tau_D^{\mathcal{C}}} e^{-(\delta+\eta+\lambda)t} \left(\delta - (1-\pi) Z_t\right) dt \right] \\
&+ \tilde{\mathbb{E}}_0 \left[\int_0^{\tau_D^{\mathcal{C}}} e^{-(\delta+\eta+\lambda)t} \mathbb{1}_{\left\{G \geq \rho(m^{\theta_M}) Z_t\right\}} \eta \left(\mathbb{1}_{\left\{Z^{\theta_M} \leq Z_D\right\}} f(Z^{\theta_M}) + \mathbb{1}_{\left\{Z^{\theta_M} > Z_D\right\}} (1-q)(1-\alpha) \right) dt \right] \\
&- \tilde{\mathbb{E}}_0 \left[\int_0^{\tau_D^{\mathcal{C}}} e^{-(\delta+\eta+\lambda)t} \mathbb{1}_{\left\{G \geq \rho(m^{\theta_M}) Z_t\right\}} \eta \rho(m^{\theta_M}) Z_t dt \right] \\
&+ \tilde{\mathbb{E}}_0 \left[\int_0^{\tau_D^{\mathcal{C}}} e^{-(\delta+\eta+\lambda)t} \lambda \mathfrak{r}(Z_t) dt \right] \\
&+ \tilde{\mathbb{E}}_0 \left[e^{-(\delta+\eta+\lambda)\tau_D^{\mathcal{C}}} (1-q)(1-\alpha) \right].
\end{aligned}
$$

with the maturity date τ^1_{mat} and the restructuring date τ^1_λ integrated out. Furthermore, if the coupon of the Markovian issuance strategy Z^{θ_M} is above the default threshold then the firm issues debt and directly defaults. This implies that the equity value is zero and the debt proceeds are the bankruptcy proceeds minus the debt issuance cost.

The firm value with both the maturity and creditor meeting jumps integrated out is given by,

$$
\begin{aligned}
f(Z_0) =& e(Z_0) + (1-q)d(Z_0) \\
=& \tilde{\mathbb{E}}_0 \left[\int_0^{\tau_D^{\mathcal{C}}} e^{-(\delta+\eta+\lambda)t} \left(\delta + (\pi - q)Z_t\right) dt \right] \\
&+ \tilde{\mathbb{E}}_0 \left[\int_0^{\tau_D^{\mathcal{C}}} e^{-(\delta+\eta+\lambda)t} \mathbb{1}_{\left\{G \geq \rho(m^{\theta_M})Z_t\right\}} \eta \left(\mathbb{1}_{\left\{Z^{\theta_M} \leq Z_D\right\}} f(Z^{\theta_M}) + \mathbb{1}_{\left\{Z^{\theta_M} > Z_D\right\}} (1-q)(1-\alpha) \right) dt \right] \\
&- \tilde{\mathbb{E}}_0 \left[\int_0^{\tau_D^{\mathcal{C}}} e^{-(\delta+\eta+\lambda)t} \mathbb{1}_{\left\{G \geq \rho(m^{\theta_M})Z_t\right\}} \eta q \rho(m^{\theta_M}) Z_t dt \right] \\
&+ \tilde{\mathbb{E}}_0 \left[\int_0^{\tau_D^{\mathcal{C}}} e^{-(\delta+\eta+\lambda)t} \mathbb{1}_{\left\{G < \rho(m^{\theta_M})Z_t\right\}} \eta (1-q)(1-\alpha) dt \right] \\
&+ \tilde{\mathbb{E}}_0 \left[\int_0^{\tau_D^{\mathcal{C}}} e^{-(\delta+\eta+\lambda)t} \lambda \sup_{a' \geq 1} \left\{ f(a' Z_t) - \mathbb{1}_{\{a'>1\}} q d(Z_t) - \mathbb{1}_{\{a'>1\}} \epsilon \right\} dt \right] \\
&+ \tilde{\mathbb{E}}_0 \left[e^{-(\delta+\eta+\lambda)\tau_D^{\mathcal{C}}} (1-q)(1-\alpha) \right].
\end{aligned}
$$

Lemma 2. *For any homogeneous and Markovian debt issuance strategy θ^M, creditor meeting intensity λ, and default strategy (Z_D, G) the firm value $f(Z)$ exists, is unique, bounded, and C^2 on $[0, Z_D]$.*

Proof. The firm value $f(Z)$ is the fixed point of the mapping A,

$$
\begin{aligned}
A(f)(Z_0) =& \tilde{\mathbb{E}}_0 \left[\int_0^{\tau_D^{\mathcal{C}}} e^{-(\delta+\eta+\lambda)t} \left(\delta + (\pi - q) Z_t\right) dt \right] \qquad (10) \\
&+ \tilde{\mathbb{E}}_0 \left[\int_0^{\tau_D^{\mathcal{C}}} e^{-(\delta+\eta+\lambda)t} \mathbb{1}_{\left\{G \geq \rho(m^{\theta_M}) Z_t\right\}} \eta \left(\mathbb{1}_{\left\{Z^{\theta_M} \leq Z_D\right\}} f(Z^{\theta_M}) + \mathbb{1}_{\left\{Z^{\theta_M} > Z_D\right\}} (1-q)(1-\alpha) \right) dt \right] \\
&- \tilde{\mathbb{E}}_0 \left[\int_0^{\tau_D^{\mathcal{C}}} e^{-(\delta+\eta+\lambda)t} \mathbb{1}_{\left\{G \geq \rho(m^{\theta_M}) Z_t\right\}} \eta q \rho(m^{\theta_M}) Z_t dt \right] \\
&+ \tilde{\mathbb{E}}_0 \left[\int_0^{\tau_D^{\mathcal{C}}} e^{-(\delta+\eta+\lambda)t} \mathbb{1}_{\left\{G < \rho(m^{\theta_M}) Z_t\right\}} \eta (1-q)(1-\alpha) dt \right] \\
&+ \tilde{\mathbb{E}}_0 \left[\int_0^{\tau_D^{\mathcal{C}}} e^{-(\delta+\eta+\lambda)t} \lambda \sup_{a' \geq 1} \left\{ f(a' Z_t) - \mathbb{1}_{\{a' > 1\}} q d(Z_t) - \mathbb{1}_{\{a' > 1\}} \epsilon \right\} dt \right] \\
&+ \tilde{\mathbb{E}}_0 \left[e^{-(\delta+\eta+\lambda)\tau_D^{\mathcal{C}}} (1-q)(1-\alpha) \right].
\end{aligned}
$$

Given a firm value $f_i(Z)$ let a_i^* solve,

$$
f_i(a_i^* Z) - \mathbb{1}_{\left\{a_i^* > 1\right\}} q d(Z) - \mathbb{1}_{\left\{a_i^* > 1\right\}} \epsilon = \sup_{a' \geq 1} \left\{ f_i(a' Z) - \mathbb{1}_{\{a' > 1\}} q d(Z) - \mathbb{1}_{\{a' > 1\}} \epsilon \right\},
$$

which implies that,

$$
\left(f_1(a_1^* Z) - \mathbb{1}_{\left\{a_1^* > 1\right\}} q d(Z) - \mathbb{1}_{\left\{a_1^* > 1\right\}} \epsilon \right) - \left(f_2(a_2^* Z) - \mathbb{1}_{\left\{a_2^* > 1\right\}} q d(Z) - \mathbb{1}_{\left\{a_2^* > 1\right\}} \epsilon \right) \leq f_1(a_1^* Z) - f_2(a_1^* Z).
$$

This implies that the mapping A is a contraction mapping given the L_∞ norm,

$$
\begin{aligned}
& A(f_1)(Z_0) - A(f_2)(Z_0) \\
& \leq \tilde{\mathbb{E}}_0 \left[\int_0^{\tau_D^{\mathcal{C}}} e^{-(\delta+\eta+\lambda)t} \left\{ \eta \mathbb{1}_{\left\{\rho(m^{\theta_M}) Z_t \leq G\right\}} \mathbb{1}_{\left\{Z^{\theta_M} \leq Z_D\right\}} \left(f_1\left(Z^{\theta_M}\right) - f_2\left(Z^{\theta_M}\right) \right) + \lambda \left(f_1(a_1^* Z_t) - f_2(a_1^* Z_t) \right) \right\} dt \right] \\
& \leq (\lambda + \eta) \tilde{\mathbb{E}}_0 \left[\int_0^{\tau_D^{\mathcal{C}}} e^{-(\delta+\eta+\lambda)t} dt \right] \| f_1 - f_2 \| \\
& \leq \frac{\lambda + \eta}{\delta + \lambda + \eta} \| f_1 - f_2 \|.
\end{aligned}
$$

The bound on the upper bound of the firm value (8) and the lower bound $-q\rho(m)Z_D/\delta$ then ensure that the fixed point of the contraction mapping is unique and a continuous function on $[0, Z_D]$. Lemma A.7 of Hugonnier et al. (2015) proves that the solution is C^2 on $[0, Z_D]$. $\square$

This shows that a solution to the model exists for any default threshold. Furthermore, at maturity the payoff of shareholders is,

$$\mathbb{1}_{\left\{G \geq \rho(m^{\theta_M})Z_t\right\}} \mathbb{1}_{\left\{Z^{\theta_M} \leq Z_D\right\}} \left(f(Z^{\theta_M}) - \rho(m^{\theta_M})Z_t\right),$$

clearly this payoff is maximized for,

$$G = \left(\mathbb{1}_{\left\{Z^{\theta_M} \leq Z_D\right\}} f(Z^{\theta_M}) + \mathbb{1}_{\left\{Z^{\theta_M} > Z_D\right\}}(1-q)(1-\alpha)\right).$$

This directly gives the optimal default strategy at maturity.

Lemma 3. *For any m, λ, and Z_D there exists a constant G such that,*

$$G = \left(\mathbb{1}_{\left\{Z^{\theta_M} \leq Z_D\right\}} f(Z^{\theta_M}) + \mathbb{1}_{\left\{Z^{\theta_M} > Z_D\right\}}(1-q)(1-\alpha)\right)$$

Proof. From Lemma A.6 of Hugonnier et al. (2015) it follows that $f(Z|m, G, Z_D)$ is continuous in G. The maximum theorem then ensures that $\sup_{Z \in [0, Z_D]} f(Z|m, G, Z_D)$ is continuous in G. Furthermore, the firm value is bounded by,

$$\left[-\frac{q\rho(m)Z_D}{\delta}, \frac{\delta + (\pi - q)Z_D}{\delta}\right],$$

for any G because of the payout at maturity and the upper bound on the firm value. The existence of the fixed point then follows from the intermediate value theorem. □

In the rest of the appendix the constant G is fixed at the optimal firm value and therefore the default decision at maturity is optimal.

C. *Restructuring*

The next step is to find the optimal restructuring strategy for any m and default threshold Z_D. Because the cash flows to the firm are non-monotonic and discontinuous at one point, see equation (6) and Fig. 8, there are two possible optimal restructuring strategies,

Proposition 3. *For any (θ_M, τ_D) such that τ_D is given by equation (4), one of the following two restructuring strategies is optimal:*

1. ***Barrier strategy**: $a > 0$ when $Z \in [0, Z_r^{bar})$ and else $a = 1$.*

2. ***Band-barrier strategy**: $a > 0$ when $Z \in [0, Z_r^{bar}) \cup (Z_r^{bnd-}, Z_r^{bnd+})$ and else $a = 1$.*

Where Z^{bar}, Z_r^{bnd-}, and Z_r^{bnd+} are thresholds at which shareholders are indifferent between restructuring and not altering the firm's capital structure. For large enough restructuring costs ϵ, $Z_r^{bar} = 0$ and shareholders never restructure the firm's debt.

Proof. Fix m and Z_D then the firm value, debt value, and restructuring surplus are $f(Z)$, $d(Z)$, and $s(Z)$. Let the thresholds Z_r^{bar} and Z^{bnd+} be,

$$\begin{aligned} Z_r^{bar} &= \sup \{ Z \geq 0 | \forall z \in [0, Z),\ s(z) > 0 \}, \\ Z_r^{bnd+} &= \inf \{ Z \leq Z_D | \forall z \in [Z, Z_D],\ s(z) = 0 \}. \end{aligned}$$

This implies that for $Z < Z_r^{bar}$ the surplus is positive and it is optimal to restructure and for $Z \geq Z_r^{bnd+}$ the surplus is zero and it is optimal postpone restructuring.[8] In some left neighbourhood of Z_r^{bar} and Z_r^{bnd+} the surplus is positive by definition. Furthermore, in this neighborhood $a^* Z > Z_r^{bar}$ and $a^* Z > Z^{bnd+}$ else the surplus would be zero since issuance cost are strictly positive. For the surplus to become zero it must be the case that $f(Z) + qd(Z)$ increases in this left neighbourhood. This implies that at Z_r^{bar} and Z_r^{bnd+} the first-order derivatives of $f(Z) + qd(Z)$ satisfy,

$$\begin{aligned} \left. \frac{\partial \left(f(Z) + qd(Z) \right)}{\partial Z} \right|_{Z = Z_r^{bar}} &\geq 0, \\ \left. \frac{\partial \left(f(Z) + qd(Z) \right)}{\partial Z} \right|_{Z = Z_r^{bnd+}} &\geq 0, \end{aligned}$$

as in Fig. 10.

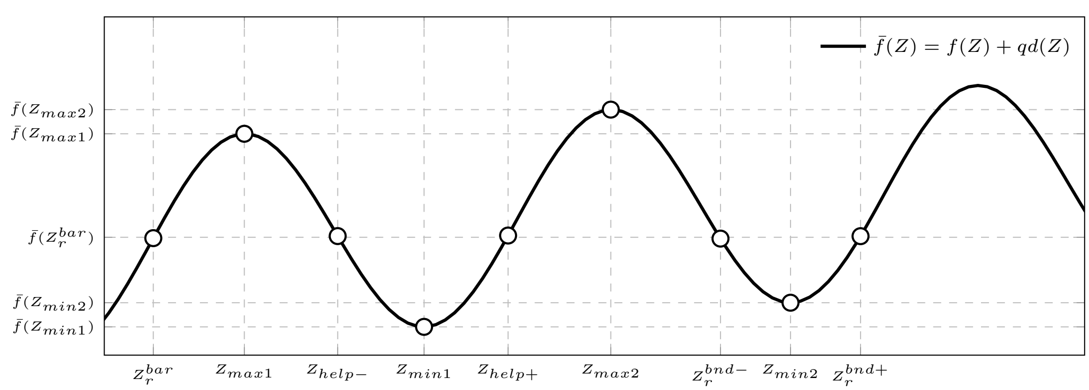


FIGURE 10: The firm value excluding current debt issuance cost $\bar{f}(Z) = f(Z) + qd(Z)$ in the contradictory case.

[8] If $Z_r^{bar} = 0$ then shareholders optimally abstain from issuing debt and the firm value boils down to the unlevered firm value of 1. Furthermore, $Z_r^{bar+} < Z_D$ because issuance cost are strictly positive at maturity and the surplus is continuous in Z.

If $Z_r^{bar} = Z_r^{bnd+}$ then the optimal strategy is a *Barrier strategy*, shareholders restructure when $Z \in [0, Z_r^{bar})$ and they meet a new creditor. The optimality follows directly from the definition of Z_r^{bar} and Z_r^{bnd+}.

In the other case, when $Z_r^{bar} < Z_r^{bnd+}$ there exists a threshold Z_r^{bnd-} such that,

$$Z_r^{bnd-} = \inf \left\{ Z < Z_r^{bnd+} | \forall z \in (Z, Z_r^{bnd+}),\ \mathcal{s}(z) > 0 \right\},$$

implying that for $Z \in (Z_r^{bnd-}, Z_r^{bnd+})$ restructuring is optimal. Using the same reasoning as before the first-order derivative of $f(Z) + qd(Z)$ satisfies,

$$\left. \frac{\partial \left(f(Z) + qd(Z) \right)}{\partial Z} \right|_{Z = Z_r^{bnd-}} \leq 0.$$

This is the *Barrier-band strategy*, shareholders restructure if $Z \in [0, Z_r^{bar}) \cup (Z_r^{bnd-}, Z_r^{bnd+})$ and they meet a new creditor. To show that this strategy is optimal it must be that case that for $Z \in [Z_r^{bar}, Z_r^{bnd-}]$ the surplus is zero,

$$\mathcal{s}(Z) = 0.$$

In the region $[0, Z_r^{bar})$ and (Z_r^{bnd-}, Z_r^{bnd+}) it is optimal to restructure by definition while for $[Z_r^{bnd+}, Z_D]$ it is optimal to postpone restructuring by definition.

Assume the surplus is positive somewhere in $[Z_r^{bar}, Z_r^{bnd-}]$, there exists a $Z \in [Z_r^{bar}, Z_r^{bnd-}]$ such that,

$$\mathcal{s}(Z) > 0.$$

Continuity of the surplus with respect to Z implies that there exist two other thresholds,

$$\begin{aligned} Z_{help-} &= \sup \left\{ Z \geq Z_r^{bar} | \forall z \in [Z_r^{bar}, Z],\ \mathcal{s}(z) = 0 \right\}, \\ Z_{help+} &= \sup \left\{ Z \geq Z_{help-} | \forall z \in (Z_{help-}, Z),\ \mathcal{s}(z) > 0 \right\}, \end{aligned}$$

with $Z_r^{bar} \leq Z_{help-} < Z_{help+} < Z_r^{bnd-}$. The situation is then as in Fig. 10.

Continuity of the surplus in Z implies that for $Z \in \{Z_r^{bar}, Z^{help-}, Z^{help+}, Z_{bnd-}, Z_{bnd+}\}$ the following holds,

$$\sup_{a'>1} f(a'Z) = f(Z) + qd(Z) + \epsilon. \tag{11}$$

Furthermore, the larger Z the smaller the parameter space of the optimization problem (Z, Z_D) and the smaller the outcome,

$$\sup_{a>1} f(aZ).$$

This lower outcome for higher Z combined with (11) implies that,

$$\begin{aligned} f(Z_r^{bar}) + qd(Z_r^{bar}) &\geq f(Z_{help-}) + qd(Z_{help-}) \\ &\geq f(Z_{help+}) + qd(Z_{help+}) \\ &\geq f(Z_r^{bnd-}) + qd(Z_r^{bnd-}) \\ &\geq f(Z_r^{bnd+}) + qd(Z_r^{bnd+}). \end{aligned} \tag{12}$$

For any $Z \in [Z_r^{bar}, Z_{help-}]$ the surplus is zero. If $a^* Z_r^{bar} < Z_{help-}$ then,

$$\begin{aligned} f(a^* Z_r^{bar}) + qd(a^* Z_r^{bar}) &> f(a^* Z_r^{bar}) \\ &= f(Z_r^{bar}) + qd(Z_r^{bar}) + \epsilon \\ &\geq f(Z_{help-}) + qd(Z_{help-}), \end{aligned}$$

and there must be a local maximum for $f(Z) + qd(Z)$ on $[Z_r^{bar}, Z_{help-}]$. If $a^* Z_r^{bar} = Z_{help-}$ then there is a contradiction because,

$$\begin{aligned} f(Z_{help-}) + qd(Z_{help-}) &= f(a^* Z_r^{bar}) + qd(a^* Z_r^{bar}) \\ &> f(a^* Z_r^{bar}) \\ &= f(Z_r^{bar}) + qd(Z_r^{bar}) + \epsilon \\ &\geq f(Z_{help-}) + qd(Z_{help-}), \end{aligned}$$

Finally, if $a^* Z_r^{bar} > Z_{help-}$ then,

$$f(Z_r^{bar}) + qd(Z_r^{bar}) = f(a^* Z_r^{bar}) - \epsilon = f(a^* Z_{help-}) - \epsilon = f(Z_{help-}) + qd(Z_{help-}),$$

and there is a local maximum as well. This implies that $[Z_r^{bar}, Z_{help-}]$ contains a local maximum Z_{max1}. Using the same reasoning $[Z_{help+}, Z_r^{bnd-}]$ contains a local maximum Z_{max2}.

For any $Z \in (Z^{help-}, Z^{help+})$ the surplus is positive and therefore,

$$f(Z) + qd(Z) \leq f(a^* Z_{help-}) - \epsilon = f(Z_{help-}) + qd(Z_{help-}) \leq f(a^* Z_{help-}).$$

Positive issuance cost $qd(Z) > 0$ then ensure that $a^* Z_{help-} > Z_{help+}$ and therefore,

$$f(Z_{help-}) + qd(Z_{help-}) = f(Z_{help+}) + qd(Z_{help+}).$$

This implies that (Z_{help-}, Z_{help+}) contains a local minimum Z_{min1}. Using the same reasoning (Z_r^{bnd-}, Z_r^{bnd+}) contains a local minimum Z_{min2}.

For $Z > \bar{Z} = f(Z^*)/\rho(m)$ shareholders default at maturity and for $Z \leq \bar{Z}$ shareholders continue to operate the firm. Since $Z_{max1} < Z_{min1} < Z_{max2} < Z_{min2}$ it must be the case that either $\bar{Z} \lesseqqgtr Z_{min1}$.

1. If $\bar{Z} \geq Z_{min1}$ let $Z_{max} = Z_{max1}$ and $Z_{min} = Z_{min1}$.

2. If $\bar{Z} > Z_{min1}$ let $Z_{max} = Z_{max2}$ and $Z_{min} = Z_{min2}$.

This implies that at maturity shareholders make the same choice at Z_{max} and Z_{min} on whether to continue to operate the firm. The kink in the cash flow of the firm value doesn't lay inside $[Z_{min}, Z_{max}]$.

Define the operator $\mathcal{L}$ as,

$$\mathcal{L}g(Z) = -(r-\delta)Z\frac{\partial g(Z)}{\partial Z} + \frac{1}{2}\sigma^2 Z^2 \frac{\partial^2 g(Z)}{\partial^2 Z}.$$

Given these results rewrite the ordinary differential equation describing $f(Z) + qd(Z)$ at Z_{min},

$$\begin{aligned}
&(\delta+\eta)\left(f(Z_{min}) + qd(Z_{min})\right) \\
&= \mathcal{L}\left(f(Z_{min}) + qd(Z_{min})\right) + \delta + \pi Z_{min} + \eta\left(f(Z^*)\mathbb{1}_{\{\rho(m)Z_{min} \leq f(Z^*)\}} + (1-\alpha)\mathbb{1}_{\{\rho(m)Z_{min} > f(Z^*)\}}\right) \\
&\quad + \lambda \mathcal{s}(Z_{min}) \\
&\geq \delta + \pi Z_{min} + \eta\left(f(Z^*)\mathbb{1}_{\{\rho(m)Z_{min} \leq f(Z^*)\}} + (1-\alpha)\mathbb{1}_{\{\rho(m)Z_{min} > f(Z^*)\}}\right) \\
&> \delta + \pi Z_{max} + \eta\left(f(Z^*)\mathbb{1}_{\{\rho(m)Z_{max} \leq f(Z^*)\}} + (1-\alpha)\mathbb{1}_{\{\rho(m)Z_{max} > f(Z^*)\}}\right) \\
&\geq \mathcal{L}\left(f(Z_{max}) + qd(Z_{max})\right) + \delta + \pi Z_{max} + \eta\left(f(Z^*)\mathbb{1}_{\{\rho(m)Z_{max} \leq f(Z^*)\}} + (1-\alpha)\mathbb{1}_{\{\rho(m)Z_{max} > f(Z^*)\}}\right) \\
&\quad + \lambda \mathcal{s}(Z_{max}) \\
&= (\delta+\eta)\left(f(Z_{max}) + qd(Z_{max})\right).
\end{aligned}$$

The first inequality follows from the non-negative surplus at Z_{min} ($\mathcal{s}(Z_{min}) \geq 0$) and the fact that Z_{min} is a local minimum ($\mathcal{L}\left(f(Z_{min}) + qd(Z_{min})\right) \geq 0$). The second inequality follows from the fact that $Z_{max} < Z_{min}$ and at maturity the payoff is independent of Z_{max} or Z_{min}. The final inequality follows from the fact that Z_{max} is a local maximum ($\mathcal{L}\left(f(Z_{max}) + qd(Z_{max})\right) \leq 0$) and the surplus is zero ($\mathcal{s}(Z_{max}) = 0$). This result contradicts the fact that,

$$\mathscr{s}(Z_{min}) \geq 0 = \mathscr{s}(Z_{max}),$$
$$\Rightarrow$$
$$f(Z_{min}) + qd(Z_{min}) \leq f(Z_{max}) + qd(Z_{max}),$$

because of the definition of the surplus (9) and the fact that $Z_{min} > Z_{max}$. This implies that it must be the case that for $Z \in [Z_{bnd}, Z_{bar-}]$ the surplus is zero $\mathscr{s}(Z) = 0$. It shows that either the barrier strategy or the barrier-band strategy is optimal. □

This lemma shows that for a given debt maturity and default threshold the optimal restructuring strategy is either a barrier strategy or a barrier-band strategy. The next step is to show that for long debt maturities the firm resorts to the barrier restructuring strategy.

Proposition 5. *For issuance strategies with a sufficiently long debt maturity the optimal restructuring strategy is a barrier strategy.*

Proof. Assume the firm issues perpetual debt $m = \infty$ $(\eta = 0)$ and a barrier-band strategy is optimal. In this case we know that for $Z_r^{bar}, Z_r^{bnd-}, Z_r^{bnd+}$ the following holds,

$$\left.\frac{\partial\left(f(Z) + qd(Z)\right)}{\partial Z}\right|_{Z=Z_r^{bar}} \geq 0,$$
$$\left.\frac{\partial\left(f(Z) + qd(Z)\right)}{\partial Z}\right|_{Z=Z_r^{bnd-}} \leq 0,$$
$$\left.\frac{\partial\left(f(Z) + qd(Z)\right)}{\partial Z}\right|_{Z=Z_r^{bnd+}} \geq 0.$$

The inequalities follow from the fact that the surplus goes from zero to positive at all these thresholds while the optimal coupon to which is restructured is neither in $[0, Z_r^{bar})$ nor in (Z_r^{bnd-}, Z_r^{bnd+}).

From equation (12) and that for $Z \in (Z_r^{bnd-}, Z_r^{bnd+})$ the firm restructure outside this interval $a^*Z > Z_r^{bnd+}$ it follows that,

$$f(Z_r^{bar}) + qd(Z_r^{bar}) \geq f(Z_r^{bnd-}) + qd(Z_r^{bnd-}) = f(Z_r^{bnd+}) + qd(Z_r^{bnd+}).$$

Furthermore, for $Z \in (Z_r^{bnd-}, Z_r^{bnd+})$ the surplus is strictly positive and therefore,

$$f(Z) + qd(Z) < f(Z_r^{bnd-}) + qd(Z_r^{bnd-}) = f(Z_r^{bnd+}) + qd(Z_r^{bnd+}).$$

These inequalities imply that there is a local maximum $\hat{Z}_{max}$ on $[Z_r^{bar}, Z_r^{bnd-})$ and local minimum $\hat{Z}_{min}$ on (Z_r^{bnd-}, Z_r^{bnd+}). For the local max'imum and minimum,

$$f(\hat{Z}_{max}) + qd(\hat{Z}_{max}) \geq f(\hat{Z}_r^{bnd-}) + qd(\hat{Z}_r^{bnd-}) > f(\hat{Z}_{min}) + qd(\hat{Z}_{min}), \tag{13}$$

and,

$$\begin{aligned}
\mathcal{L}\left(f(\hat{Z}_{min}) + qd(\hat{Z}_{min})\right) \geq 0, \\
\mathcal{L}\left(f(\hat{Z}_{max}) + qd(\hat{Z}_{max})\right) \leq 0.
\end{aligned}$$

Similar to the prove of the previous proposition,

$$\begin{aligned}
&(\delta + \eta)\left(f(\hat{Z}_{min}) + qd(\hat{Z}_{min})\right) \\
&= \mathcal{L}\left(f(\hat{Z}_{min}) + qd(\hat{Z}_{min})\right) + \delta + \pi\hat{Z}_{min} + \lambda s(\hat{Z}_{min}) \\
&> \delta + \pi\hat{Z}_{min} \\
&> \delta + \pi\hat{Z}_{max} \\
&\geq \mathcal{L}\left(f(\hat{Z}_{max}) + qd(\hat{Z}_{max})\right) + \delta + \pi\hat{Z}_{max} + \lambda s(\hat{Z}_{max}) \\
&= (\delta + \eta)\left(f(\hat{Z}_{max}) + qd(\hat{Z}_{max})\right).
\end{aligned}$$

The first inequality follows from the fact that $\hat{Z}_{min}$ is a local minimum and the surplus is positive. The fact that $\hat{Z}_{min} > Z_r^{bnd-} > \hat{Z}_{max}$ ensures the second inequality. The fact that $\hat{Z}_{max}$ is a local maximum and the surplus is zero implies the last inequality. This contradicts the before found inequality (13). It shows that for perpetual debt the optimal restructuring strategy is a barrier strategy.

Furthermore, for $\eta = 0$ shareholders are never indifferent between restructuring or not,

$$\nexists Z > Z_r^{bar} \sup_{a' \geq 1} f(a'Z) - f(Z) - qd(Z) - \epsilon = 0.$$

Assume there is a $Z > Z_r^{bar}$ for which shareholders are indifferent. Let $\hat{Z}_{min} < Z_D$ be the largest indifference point. This indifference point is a local minimum for $f(Z) + qd(Z)$ since for $Z > Z_r^{bar}$ the surplus is zero and for $Z \in (\hat{Z}_{min}, Z_D)$,

$$\sup_{a' \geq 1} f(a'Z) - f(Z) - qd(Z) - \epsilon < 0,$$

by definition of $\hat{Z}_{min}$.[9] Both at $\hat{Z}_{min}$ and Z_r^{bar} it holds that,

$$\sup_{a>1} f(aZ) = f(Z) + qd(Z) + \epsilon.$$

Furthermore, the larger Z the smaller the parameter space of the optimization problem (Z, Z_D) and the smaller the outcome,

$$\sup_{a>1} f(aZ).$$

This implies that,

$$f(\hat{Z}_r^{bar}) + qd(\hat{Z}_r^{bar}) \geq f(\hat{Z}_{min}) + qd(\hat{Z}_{min}).$$

Define $\hat{Z}_0$ as,

$$\hat{Z}_0 = \sup\left\{Z \in [Z_r^{bar}, \hat{Z}_{min}] | \forall z \in [Z_r^{bar}, Z],\ \sup_{a>1} f(aZ) - f(Z) - qd(Z) - \epsilon = 0\right\},$$

the largest point above Z_r^{bar} such that the surplus of restructuring is zero for $Z \in [Z_r^{bar}, \hat{Z}_0]$. There are now two cases:

- If $\hat{Z}_0 > Z_r^{bar}$ then $\hat{Z}_{max} = Z_r^{bar}$ is a local maximum with $\hat{Z}_{max} < Z_{min}$.
- If $\hat{Z}_0 = Z_r^{bar}$ then at Z_r^{bar} the first-order derivative of $f(Z) + qd(Z)$ satisfies,

$$\frac{\partial(f(Z) + qd(Z))}{\partial Z} > 0.$$

  Because this function is continuously differentiable there exists a local maximum $\hat{Z}_{max} \in (Z_r^{bar}, \hat{Z}_{min})$

For $\hat{Z}_{max}$ it holds that,

$$f(\hat{Z}_{max}) + qd(\hat{Z}_{max}) \geq f(\hat{Z}_r^{bar}) + qd(\hat{Z}_r^{bar}) \geq f(\hat{Z}_{min}) + qd(\hat{Z}_{min}).$$

[9]If in some left neighborhood of $\hat{Z}_{min}$ the surplus of restructuring is zero then it must be the case that the optimizer a^* causes $a^*Z > \hat{Z}_{min}$, which ensures that $\hat{Z}_{min}$ is a local minimum for $f(Z) + qd(Z)$. Assume this is not the case for $\bar{Z}$ and let $\bar{a}^*$ be the optimizer. For $\bar{a}^*\bar{Z} \in (\bar{Z}, \hat{Z}_{min}]$ it then holds that,

$$\sup_{a' \geq 1} f(a'\bar{a}^*\bar{Z}) - f(\bar{a}^*\bar{Z}) - qd(\bar{a}^*\bar{Z}) - \epsilon = -qd(\bar{a}^*\bar{Z}) - \epsilon < 0,$$

which contradicts the fact that the surplus of restructuring is zero in $[\bar{Z}, \hat{Z}_{min}]$.

Similar arguments as before lead to a contradiction,

$$\begin{aligned}
&(\delta+\eta)\left(f(\hat{Z}_{min})+qd(\hat{Z}_{min})\right)\\
&=\mathcal{L}\left(f(\hat{Z}_{min})+qd(\hat{Z}_{min})\right)+\delta+\pi\hat{Z}_{min}+\lambda\jmath(\hat{Z}_{min})\\
&\geq\delta+\pi\hat{Z}_{min}\\
&>\delta+\pi\hat{Z}_{max}\\
&\geq\mathcal{L}\left(f(\hat{Z}_{max})+qd(\hat{Z}_{max})\right)+\delta+\pi\hat{Z}_{max}+\lambda\jmath(\hat{Z}_{max})\\
&=(\delta+\eta)\left(f(\hat{Z}_{max})+qd(\hat{Z}_{max})\right).
\end{aligned}$$

Therefore, for $\eta=0$ shareholders are never indifferent for $Z>Z_r^{bar}$.

From Lemma A.6 of Hugonnier et al. (2015) it follows that the firm value both $f(Z)$ and $f(Z)+qd(Z)$ are continuous in η. Observe that for $\eta=0$ the firm and debt value are independent of G. Therefore $\sup_{a'\geq 1} f(a'Z)-f(Z)-qd(Z)-\epsilon$ is continuous in η for any G. This ensures that for η close to zero, long maturity debt $m=1/\eta$, it is optimal to use the barrier restructuring strategy. $\square$

D. Default Threshold

The next step is to show that the optimal firm value is finite and therefore the optimal default barrier Z_D is finite. In this section $\rho(m)$ is replaced by $\rho(1/\eta)$ to simplify notation.

Lemma 4. *For any m and λ the optimal firm value $f(Z^*)$ given the equity value maximizing default strategy is finite if,*

$$\pi<\begin{cases}1 & \eta\geq(\pi-q)/\rho(1/\eta)q\\ \frac{1+\eta\rho(1/\eta)+\eta\frac{(q+q\eta\rho(1/\eta))}{\delta}}{1+\frac{\eta}{\delta}} & \eta<(\pi-q)/\rho(1/\eta)q\end{cases}.$$

Furthermore, given $\pi<1$ for,

$$\eta>\max\left(\frac{\pi-q}{\rho(1/\eta)q},\frac{\delta(\pi-q)}{(1-\pi)(\alpha+q(1-\alpha))}\right),$$

shareholders optimally abstain from issuing debt.

Proof. The stochastic process that describes the evolution of Z_t is a geometric Brownian motion, that is independent of the maturity date π^1_{mat}. Shareholders and creditors agree on the ex ante distribution and ex post outcomes of the stochastic process Z_t. What if instead of fixing the stochastic process for Z_t exogenously shareholders can pick ex ante any time-homogeneous stochastic process, that is independent of the debt maturity date? This extra degree of freedom causes the optimal firm value to dominate the firm value in the original model.

Because the stochastic process needs to be time-homogeneous and independent of the maturity date the model is still time-homogeneous. Make the following two observations:

1. Default is inefficient since a fraction α of the unlevered firm value is lost.

2. The tax benefits on the coupon payments net off issuance cost are strictly positive $\pi > q$. This gives incentive to increase the coupon.

Any uncertainty in the stochastic process increases the probability of inefficient default or causes the firm to not fully utilize its debt capacity. Therefore, the optimal process is a deterministic process. Non-constant processes Z_t are either strictly below or above the optimal default threshold at some point. In the first case the full debt capacity of the firm isn't used while in the second case there is inefficient default. For any fixed m it is clear that the optimal solution is a constant Z_t,

$$dZ_t = 0.$$

With this stochastic process the tax benefits of debt are maximized while inefficient default is prevented. This implies that the only uncertainty that remains in the model comes from the debt maturity date π^1_{mat}. Furthermore, for a constant Z_t the option to restructure the debt is worthless.

Depending on the debt maturity issuing debt is positive or negative NPV:

1. If $\eta > (\pi - q)/\rho(1/\eta)q$ issuing debt is negative NPV, if the firm never defaults at maturity, because the issuance cost over both the coupon and principal repayments dominate the tax benefits,

$$\pi - q - \eta\rho(1/\eta)q < 0.$$

   In this case shareholders pick one of the following two strategies:

   (a) The firm issues no debt and the firm value is 1.

   (b) The firm issues debt and defaults at maturity (because shareholders cannot foresee when the debt is going to mature). In this case the equity value is given by,

$$\frac{\delta - (1 - \pi)Z_c}{\delta + \eta}.$$

   The equity value is the operating income minus the coupon payments until the debt matures. Because the equity value should be non-negative the optimal default threshold is $Z_D = \delta/(1-\pi)$ and therefore the optimal firm value,

$$\frac{\delta + (\pi - q)Z_D + \eta(1-q)(1-\alpha)}{\delta + \eta} = \frac{\delta\left(1 + \frac{\pi - q}{1-\pi}\right) + \eta(1-q)(1-\alpha)}{\delta + \eta} < \infty.$$

2. If $\eta < (\pi - q)/\rho(1/\eta)q$ issuing debt is positive NPV because the tax benefits exceed the issuance cost and the firm value for a given strategy Z_c and no default at maturity is,

$$\frac{\delta + (\pi - q - q\eta\rho(1/\eta))Z_c}{\delta}.$$

This implies that for any constant control the equity value is,

$$\frac{\delta - (1 - \pi + \eta\rho(1/\eta))Z_c + \eta\frac{\delta + (\pi - q - q\eta\rho(1/\eta))Z_c}{\delta}}{\delta + \eta}.$$

At the optimal default threshold Z_D the equity value is zero and for $Z_c < Z_D$ the equity value is positive. For this to be the case it must be that,

$$0 > \frac{1}{\delta + \eta}\left(-(1 - \pi + \eta\rho(1/\eta)) + \eta\frac{\pi - q - q\eta\rho(1/\eta)}{\delta}\right),$$

which is the derivative of the equity value with respect to Z_c. In this case increasing Z_c decreases the equity value and at some point shareholders default.

This derivative is negative for

$$\begin{aligned} 0 &< 1 - \pi + \eta\rho(1/\eta) - \eta\frac{(\pi - q - q\eta\rho(1/\eta))}{\delta}, \\ \pi\left(1 + \frac{\eta}{\delta}\right) &< 1 + \eta\rho(1/\eta) + \eta\frac{(q + q\eta\rho(1/\eta))}{\delta}, \\ \pi &< \frac{1 + \eta\rho(1/\eta) + \eta\frac{(q + q\eta\rho(1/\eta))}{\delta}}{1 + \frac{\eta}{\delta}}. \end{aligned}$$

Because Z_c is a constant there is no added value to debt restructuring and the optimal firm value when shareholders are allowed to restructure upwards is also finite. This implies the first part of the lemma.

For the second part observe that, for η bigger than the boundary specified in the lemma the first case applies. Furthermore, abstaining from issuing debt leads to the upper bound on the firm value. This upper bound can be attained in the model by abstaining from issuing debt. □

Proposition 1. *The equilibrium debt maturity m^* from Theorem 1 is positive $m^* > \bar{m} > 0$,* *where the constant $\bar{m}$ is defined in the appendix.*

Proof. The result follows directly from Lemma 4. For $\pi < 1$ and debt maturities shorter than,

$$\eta > \max\left(\frac{\pi - q}{\rho(1/\eta)q}, \frac{\delta(\pi - q)}{(1-\pi)(\alpha + q(1-\alpha))}\right),$$
$$m < \bar{m} = \left(\max\left(\frac{\pi - q}{\rho(1/\eta)q}, \frac{\delta(\pi - q)}{(1-\pi)(\alpha + q(1-\alpha))}\right)\right)^{-1},$$

the firm value is one and shareholders optimally abstain from issuing debt. If shareholders issue debt is must be that the debt maturity is longer than $\bar{m}$. □

The second step is showing that given the finite firm value there exists a finite optimal default threshold. For simplicity the optimal default time τ_D is replaced by the default threshold Z_D. Optimal default at maturity is already incorporated because of Lemma 3.

Lemma 5. *For $\pi < \pi^*(\pi)$ there exists a finite default threshold that satisfies the smooth pasting condition for any θ_M and one-shot first period deviation from θ_M. Under $\lambda = 0$ or Assumption 1 the default threshold for θ_M or for a one-shit first period deviation of θ_M leads to the optimal default time of the equity value.*

Proof. The proof is setup as follows,

1. First, there exists an optimal default threshold Z_D that satisfies the smooth pasting condition assuming that creditors correctly conjecture the default threshold.

2. Fix the default threshold for the optimal firm value $o(\theta_M, a, \tau_D)$ and debt value $d(Z)$ at the default threshold Z_D found before. The current debt issuance strategy $\hat{\eta}$ and $\hat{Z}$ is a one-shot first period deviation from θ_M. Let $\hat{Z}_D$ be the default threshold for the current issue of debt outstanding. There exists a $\hat{Z}_D$ that satisfies the smooth pasting condition.

3. The default threshold $\hat{Z}_D$ that follows from the smooth pasting condition satisfies the HJB and therefore leads to the optimal stopping time of the equity value. Setting $\hat{\eta} = \eta^{\theta_M}$, $\hat{Z} = Z^{\theta_M}$, and $\hat{Z}_D = Z_D$ leads to the result for the θ_M issuance strategy.

In this proof the equity value is written as,

$$e(Z|Z_D) = e(Z, m^{\theta_M}|\theta_M, a, \tau_D),$$

with τ_D defined by (G, Z_D) and G set equal to the optimal firm value $o(\theta_M, a, \tau_D)$. The optimal firm value $o(\theta_M, a, \tau_D)$, debt value $d(Z)$, and surplus $s(Z)$ are defined as before.

1. From Lemma 4 it follows that the firm value is bounded from above by some constant H for any m and $\pi < \pi^*(\pi)$. The cash flow of the equity value is (strictly) bounded from above by,

$$\delta - (1-\pi)Z + \eta(o(\theta_M, a, \tau_D) - \rho(1/\eta)Z)^+ + \lambda s(Z) \leq \delta - (1-\pi)Z + (\eta+\lambda)H.$$

This implies that given some default threshold Z_D for $Z \in [0, Z_D)$ the equity value is strictly bounded from above by,

$$e(Z|Z_D) < \tilde{\mathbb{E}}_0\left[\int_0^{\tau_D^{\mathcal{C}}} e^{-(r+\eta+\lambda)t}\left(\delta - (1-\pi)Z + (\eta+\lambda)H\right)dt\right],$$

where $\tau_D^{\mathcal{C}} = \inf\{t > 0 | Z_t \geq Z_D\}$.

The optimal stopping time of the bound on the equity value is a standard optimal stopping problem. The solution is a barrier default strategy with a finite optimal default threshold $Z_D^{help} < \infty$ that follows from the smooth pasting condition, see Dayanik and Karatzas (2003). In case $\tau_D^{\mathcal{C}}$ is given by the default threshold Z_D^{help} the smooth pasting condition of the equity value is non-negative,

$$\left.\frac{\partial e(Z|Z_D^{help})}{\partial Z}\right|_{Z=Z_D^{help}} \geq 0.$$

Furthermore, at $Z = 0$ the cash flow of the equity value is strictly positive and default is suboptimal therefore,

$$\left.\frac{\partial e(Z|0)}{\partial Z}\right|_{Z=0} \leq 0.$$

From Hugonnier et al. (2015), Lemma A.6 it follows that the first-order derivative of the equity value at the default threshold is continuous in the default threshold. The intermediate value theorem implies that there exists a default threshold $Z_D \in [0, Z_D^{help}]$ that satisfies the smooth pasting condition,

$$Z_D = \inf\left\{\tilde{Z}_D > 0 \middle| \left.\frac{\partial e(Z|\tilde{Z}_D)}{\partial Z}\right|_{Z=\tilde{Z}_D} = 0\right\}.$$

By definition $Z_D < Z_D^{help} < \infty$ is finite.

2. Fix the default threshold for the optimal firm value $o(\theta_M, a, \tau_D)$ using the default threshold Z_D found before. Assume there is a one-shot first period deviation of θ_M, $\hat{\eta}$ and $\hat{Z}$.

The equity value for this one-shot deviation given a default threshold $\hat{Z}_D$ is $\hat{e}(Z|\hat{Z}_D)$,

$$\begin{aligned}
&\hat{e}(Z|\hat{Z}_D) \\
&= \tilde{\mathbb{E}}_0\left[\int_0^{\tau_{\hat{Z}_D}} e^{-(\delta+\hat{\eta}+\lambda)t}\left(\delta-(1-\pi)Z_t+\hat{\eta}(o(\theta_M,a,\tau_D)-\rho(1/\hat{\eta})Z_t)^+\right)dt\right] \\
&+\tilde{\mathbb{E}}_0\left[\int_0^{\tau_{\hat{Z}_D}} e^{-(\delta+\hat{\eta}+\lambda)t}\lambda\sup_{a'\geq 1}\left\{\hat{e}(a'Z_t|\hat{Z}_D)+(1-q\mathbb{1}_{\{a'>1\}})\hat{d}(a'Z_t|\hat{Z}_D)-\mathbb{1}_{\{a'>1\}}\epsilon\right\}dt\right] \\
&-\tilde{\mathbb{E}}_0\left[\int_0^{\tau_{\hat{Z}_D}} e^{-(\delta+\hat{\eta}+\lambda)t}\lambda\hat{d}(Z_t|\hat{Z}_D)dt\right],
\end{aligned}$$

where $\tau_{\hat{Z}_D}=\inf\{t>0|Z_t\geq\hat{Z}_D\}$ and $\hat{d}(Z|\hat{Z}_D)=d(Z,1/\hat{\eta}|\hat{Z}_D,o(\theta_M,a,\tau_D))$. Similar arguments as for Z_D show there exists a $\hat{Z}_D<Z_D^{help}$ that satisfies the smooth pasting condition.

3. The equity value $\hat{e}(Z|\hat{Z}_D)$ is piecewise C^2, therefore showing optimality of the default threshold $\hat{Z}_D$ boils down to showing that the HJB is satisfied.

   - The equity value function is non-negative for $Z\leq\hat{Z}_D$. For $\lambda=0$ we know the following about the equity value,

     (a) The derivative is negative in some left neighborhood of $\hat{Z}_D$. This follows from fact that the smooth pasting condition holds at $\hat{Z}_D$ and the derivative of the cash flow is negative. It can be shown by differentiating the ordinary differential equation describing the equity value and using the smooth pasting condition at $\hat{Z}_D$ as one of the boundary conditions. For Z arbitrarily close to $\hat{Z}_D$ the first-order derivative is negative.

     (b) For $Z=0$ the equity value is positive,

     $$\hat{e}(0|\hat{Z}_D)=\frac{\delta+\eta(o(\theta_M,\tau_D))^+}{\delta+\eta}>0.$$

     (c) The equity value function is C^2 since the cash flows are continuous. This follows from the ordinary differential equation describing the equity value.

     (d) Assume the equity value is negative somewhere on $[0,\hat{Z}_D]$. From (a), (b), and (c) it follows that the local minimum and maximum $\hat{Z}_{min}$ and $\hat{Z}_{max}$ exist. Furthermore, these extremes satisfy,

     $$\begin{aligned}
     \hat{e}(\hat{Z}_{min}|\hat{Z}_D)&>0,\\
     \hat{e}(\hat{Z}_{max}|\hat{Z}_D)&<0,
     \end{aligned}$$

and $\hat{Z}_{min} < \hat{Z}_{max}$. The equity value for this case is depicted in Fig. 11.

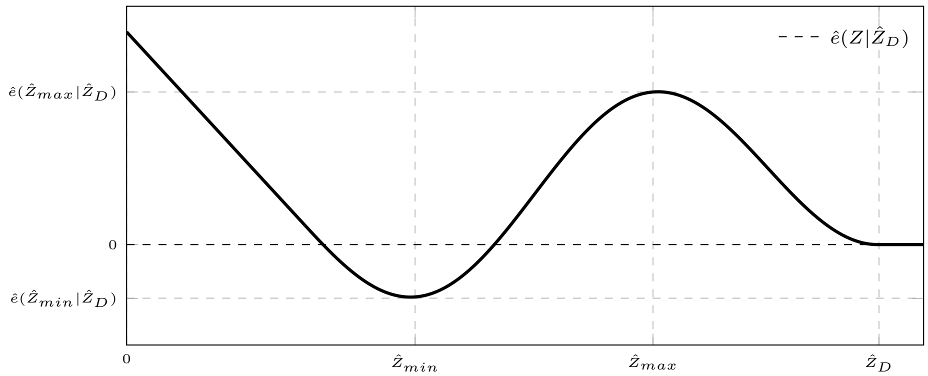


FIGURE 11: Equity value that becomes negative at some point on $[0, \hat{Z}_D]$.

(e) From the ordinary differential equation describing the equity value,

$$(\delta + \hat{\eta})\hat{e}(Z|\hat{Z}_D) = \mathcal{L}\hat{e}(Z|\hat{Z}_D) + \delta - (1-\pi)Z + \hat{\eta}(o(\theta_M, a, \tau_D) - \rho(1/\hat{\eta})Z)^+,$$

it follows that the cash flow for $\hat{Z}_{min}$ and $\hat{Z}_{max}$ satisfies,

$$\begin{aligned}
&\delta - (1-\pi)\hat{Z}_{min} + \hat{\eta}(o(\theta_M, a, \tau_D) - \rho(1/\hat{\eta})\hat{Z}_{min})^+ \\
&= (\delta + \hat{\eta})\hat{e}(\hat{Z}_{min}|\hat{Z}_D) - \mathcal{L}\hat{e}(\hat{Z}_{min}|\hat{Z}_D) < 0, \\
&\delta - (1-\pi)\hat{Z}_{max} + \hat{\eta}(o(\theta_M, a, \tau_D) - \rho(1/\hat{\eta})\hat{Z}_{max})^+ \\
&= (\delta + \hat{\eta})\hat{e}(\hat{Z}_{max}|\hat{Z}_D) - \mathcal{L}\hat{e}(\hat{Z}_{max}|\hat{Z}_D) > 0.
\end{aligned}$$

This contradicts the fact that the cash flow to shareholders is decreasing in Z and $\hat{Z}_{min} < \hat{Z}_{max}$.

This result ensures that the equity value is non-negative on $[0, \hat{Z}_D)$ for $\lambda = 0$. For Assumption 1 the non-negativity of the equity value on $[0, \hat{Z}_D)$ follows directly from the assumption.

- The equity value function is zero for $Z \geq \hat{Z}_D$ by definition.
- For $Z \in [\hat{Z}_D, \infty)$ the equity value is zero and therefore the cash flow with debt

maturity and restructuring integrated out is,

$$
\begin{aligned}
&\delta - (1-\pi)Z + \hat{\eta}(o(\theta_M, a, \tau_D) - \rho(1/\hat{\eta})Z)^+ \\
&\quad + \lambda \sup_{a' \geq 1} \left\{ \hat{e}(a'Z|\hat{Z}_D) + (1 - q\mathbb{1}_{\{a'>1\}})\hat{d}(a'Z|\hat{Z}_D) - \hat{d}(Z|\hat{Z}_D) - \mathbb{1}_{\{a'>1\}}\epsilon \right\} \\
=&\delta - (1-\pi)Z + \hat{\eta}(o(\theta_M, a, \tau_D) - \rho(1/\hat{\eta})Z)^+ \\
&\quad + \lambda \sup_{a' \geq 1} \left\{ (1 - q\mathbb{1}_{\{a'>1\}})(1-\alpha) - (1-\alpha) - \mathbb{1}_{\{a'>1\}}\epsilon \right\}.
\end{aligned}
$$

This cash flow is decreasing in Z. At $\hat{Z}_D$ the value matching and smooth pasting condition are satisfied and in some left neighborhood the equity value is positive, which implies that the second-order derivative is non-negative because the equity value is C^2. From the ordinary differential equation describing the equity value,

$$
\begin{aligned}
0 =& -(\delta + \hat{\eta} + \lambda)\hat{e}(Z|\hat{Z}_D) + \mathcal{L}\hat{e}(Z|\hat{Z}_D) \\
&+ \delta - (1-\pi)Z + \hat{\eta}(o(\theta_M, a, \tau_D) - \rho(1/\hat{\eta})Z)^+ \\
&+ \lambda \sup_{a' \geq 1} \left\{ \hat{e}(a'Z|\hat{Z}_D) + (1 - q\mathbb{1}_{\{a'>1\}})\hat{d}(a'Z|\hat{Z}_D) - \hat{d}(Z|\hat{Z}_D) - \mathbb{1}_{\{a'>1\}}\epsilon \right\},
\end{aligned}
\tag{14}
$$

it follows that at $\hat{Z}_D$ the cash flow is non-positive,

$$
\begin{aligned}
&\frac{1}{2}\sigma^2 \frac{\partial^2 \hat{e}(Z|\hat{Z}_D)}{\partial^2 Z}\bigg|_{Z=\hat{Z}_D} \\
&= -\left(\delta - (1-\pi)\hat{Z}_D + \hat{\eta}\left(o(\theta_M, a, \tau_D) - \rho(1/\hat{\eta})\hat{Z}_D\right)^+\right) \\
&\quad - \lambda \sup_{a' \geq 1} \left\{ (1 - q\mathbb{1}_{\{a'>1\}})(1-\alpha) - (1-\alpha) - \mathbb{1}_{\{a'>1\}}\epsilon \right\} \\
&\geq 0.
\end{aligned}
$$

This ensures that the cash flow is non-positive for $Z \geq \hat{Z}_D$ and the ordinary differential equation describing the equity value (14) is non-positive.

- For $Z \in [0, \hat{Z}_D)$ the equity value satisfies the above given ordinary differential equation (14).

The equity value satisfies the HJB equation everywhere,

$$
\begin{aligned}
0 = \max \Big\{ & -\hat{e}(Z|\hat{Z}_D), -(\delta + \hat{\eta} + \lambda)\hat{e}(Z|\hat{Z}_D) \\
& + \mathcal{L}\hat{e}(Z|\hat{Z}_D) + \delta - (1-\pi)Z + \hat{\eta}(o(\theta_M, a, \tau_D) - \rho(1/\hat{\eta})Z)^+ \\
& + \lambda \sup_{a' \geq 1} \left\{ \hat{e}(a'Z|\hat{Z}_D) + (1 - q\mathbb{1}_{\{a'>1\}})\hat{d}(a'Z|\hat{Z}_D) - \hat{d}(Z|\hat{Z}_D) - \mathbb{1}_{\{a'>1\}}\epsilon \right\} \Big\},
\end{aligned}
$$

standard verification arguments, as in Dayanik and Karatzas (2003), then imply that $\hat{Z}_D$ leads to the optimal stopping time for the one-shot deviation. Setting $\hat{\eta} = \eta^{\theta_M}$, $\hat{Z} = Z^{\theta_M}$, and $\hat{Z}_D = Z_D$ leads to the result for the θ_M issuance strategy.

□

E. *Original Model*

Theorem 1. *For $\pi < \pi^*(\pi)$ there exists a Markov Perfect Equilibrium. In this equilibrium,*

- *The equilibrium issuance strategy θ_M^* maximizes the firm value assuming shareholders default optimally.*
- *The equilibrium default strategy τ_D^* is,*

$$\tau_D^* = \inf \{\mathcal{C}, \mathcal{P}\},$$

where the default barrier Z_D^* satisfies the smooth pasting condition.

- *The equilibrium debt price equals the expected discounted cash flows of the debt given τ_D^*.*
- *Off-equilibrium, creditors belief that any deviation from the optimal issuance strategy is a one-shot deviation. The off-equilibrium debt price is the expected discounted cash flows of the debt given the optimal default strategy that follows from these beliefs.*

Proof. Take $\theta_M^* = \arg\sup_{\theta_M \in \mathbb{R}_+^2} o(\theta_M, \tau_D)$. From Lemma 4 it follows that if $\eta^{\theta_M^*} \leq \bar{\eta}$ then the optimal firm value is equal to the unlevered firm value 1. Therefore the optimal debt maturity $\eta^{\theta_M^*} < \infty$ is attained. The optimality and existence of a finite barrier default strategy follows from Lemma 5. The existence of a finite default barrier implies that $Z^{\theta_M^*} < \infty$ is attained.

Because of the time-homogeneity of the model it is sufficient to look at one-shot first period deviations to prove existence of a Markov Perfect Equilibrium. Shareholders deviate from

θ_M in the first period with $\hat{Z} > 0$ and $\hat{m}$. Lemma 5 shows that the optimal default strategy is given by (4) with the finite default threshold satisfying the smooth pasting condition. The proof now has several steps:

1. Assume there exists a one-shot deviation $(\hat{Z}, \hat{m})$, with the optimal default strategy given by $\hat{Z}_D$ and $\hat{G} = o(\theta_M^*, \tau_D)$, that improves the firm value. Observe that firm value $\hat{f}(Z)$ at $\hat{Z}$ in this case is equal to,

$$\begin{aligned}
\hat{f}(\hat{Z}_0|\hat{Z}_D, \hat{G}) =& \tilde{\mathbb{E}}_0\left[\int_0^{\tau_{\hat{Z}_D}} e^{-(\delta+\hat{\eta})t}\left(\delta + (\pi - q)Z_t\right) dt\right] \\
&+ \tilde{\mathbb{E}}_0\left[\int_0^{\tau_{\hat{Z}_D}} e^{-(\delta+\hat{\eta})t}\mathbb{1}_{\{\hat{G}\geq\rho(1/\hat{\eta})Z_t\}}\hat{\eta}\left(o(\theta_M^*, \tau_D^*) - q\rho(1/\hat{\eta})Z_t\right) dt\right] \\
&+ \tilde{\mathbb{E}}_0\left[\int_0^{\tau_{\hat{Z}_D}} e^{-(\delta+\hat{\eta})t}\mathbb{1}_{\{\hat{G}<\rho(1/\hat{\eta})Z_t\}}\hat{\eta}(1-q)(1-\alpha)dt\right] \\
&+ \tilde{\mathbb{E}}_0\left[e^{-(\delta+\hat{\eta})\tau_{\hat{Z}_D}}(1-q)(1-\alpha)\right] \\
&> o(\theta_M^*, \tau_D^*).
\end{aligned}$$

2. The mapping K,

$$\begin{aligned}
K(\hat{f})(Z_0|\hat{Z}_D, \hat{G}) =& \tilde{\mathbb{E}}_0\left[\int_0^{\tau_{\hat{Z}_D}} e^{-(\delta+\hat{\eta})t}\left(\delta + (\pi - q)Z_t\right) dt\right] \\
&+ \tilde{\mathbb{E}}_0\left[\int_0^{\tau_{\hat{Z}_D}} e^{-(\delta+\hat{\eta})t}\mathbb{1}_{\{\hat{G}\geq\rho(1/\hat{\eta})Z_t\}}\hat{\eta}\left(\hat{f}(\hat{Z}) - q\rho(1/\hat{\eta})Z_t\right) dt\right] \\
&+ \tilde{\mathbb{E}}_0\left[\int_0^{\tau_{\hat{Z}_D}} e^{-(\delta+\hat{\eta})t}\mathbb{1}_{\{\hat{G}<\rho(1/\hat{\eta})Z_t\}}\hat{\eta}(1-q)(1-\alpha)dt\right] \\
&+ \tilde{\mathbb{E}}_0\left[e^{-(\delta+\hat{\eta})\tau_{\hat{Z}_D}}(1-q)(1-\alpha)\right],
\end{aligned}$$

is a contraction mapping just as A (10) in Lemma 2. Furthermore, this mapping is increasing in $\hat{f}(\hat{Z})$ and therefore,

$$\begin{aligned}
K^\infty(\hat{f})(\hat{Z}|\hat{Z}_D, \hat{G}) &\geq K^n(\hat{f})(\hat{Z}|\hat{Z}_D, \hat{G}) \\
&\geq K^{n-1}(\hat{f})(\hat{Z}|\hat{Z}_D, \hat{G}) \\
&\geq K(\hat{f})(\hat{Z}|\hat{Z}_D, \hat{G}) \\
&> o(\theta_M^*, \tau_D^*) \\
&\geq 1,
\end{aligned} \tag{15}$$

where K^n is the mapping K applied n times to $\hat{f}$. The fixed point of this mapping is the solution to the firm value given issuance strategy $\hat{\theta}_M = (\hat{Z}, \hat{m})$ and default strategy $(\hat{Z}_D, \hat{G})$.

3. The intermediate value theorem combined with Lemma A.6 of Hugonnier et al. (2015) implies that there exists an optimal $G \in \left(o(\theta_M^*, \tau_D^*), \frac{\delta + (\pi - q)\hat{Z}_D}{\delta} \right]$ that solves solves,

$$G = K^{\infty}(\hat{f})(\hat{Z}|G, \hat{Z}_D).$$

   Let $\underset{\smile}{G}$ be this fixed point then,

$$K^{\infty}(\hat{f})(\hat{Z}|\underset{\smile}{G}, \hat{Z}_D) = \underset{\smile}{G} > o(\theta_M^*, \tau_D^*).$$

4. The cash flows of the equity value satisfy,

$$\delta - (1-\pi)Z + \hat{\eta}\left(o(\theta_M^*, \tau_D^*) - \rho(1/\hat{\eta})Z\right)^+ \leq \delta - (1-\pi)Z + \hat{\eta}\mathbb{1}_{\left\{o(\theta_M^*, \tau_D^*) \geq \rho(1/\hat{\eta})Z\right\}}\left(\underset{\smile}{G} - \rho(1/\hat{\eta})Z\right)$$

$$\leq \delta - (1-\pi)Z + \hat{\eta}\left(\underset{\smile}{G} - \rho(1/\hat{\eta})Z\right)^+$$

   The equity value given the strategy $\hat{\theta}_M$ dominates the equity value of the one shot deviation given by $(\hat{Z}, \hat{m})$ therefore at the default threshold,

$$\left.\frac{\partial e(Z, \hat{m}|\hat{\theta}_M, \underset{\smile}{G}, \hat{Z}_D)}{\partial Z}\right|_{Z=\hat{Z}_D} \leq 0.$$

   Furthermore, by definition of Z_D^{help} it must be that $\hat{Z}_D < Z_D^{help}$ therefore between $[\hat{Z}_D, Z_D^{help}]$ there exists another default threshold that satisfies the smooth pasting condition $\underset{\smile}{Z}_D$.

5. The firm value is increasing in the default threshold since,

$$\begin{aligned}
&\delta + (\pi - q)\hat{Z}_D + \hat{\eta}(1-q)(1-a) > (\delta + \hat{\eta})(1-q)(1-\alpha), \\
&\delta + (\pi - q)\hat{Z}_D + \hat{\eta}\left(f(\hat{Z}|\hat{\theta}_M, \hat{G}, \hat{Z}_D) - q\rho(1/\hat{\eta})\right)\hat{Z}_D > (\delta + \hat{\eta})(1-q)(1-\alpha).
\end{aligned}$$

   For the second case observe that if $(\pi - q - \hat{\eta}q\rho(1/\hat{\eta})) > 0$ the results holds. If $(\pi - q - \hat{\eta}q\rho(1/\hat{\eta})) < 0$, because shareholders never default at maturity for any trajectory of Z_t issuing debt is negative NPV. The firm value is strictly smaller than 1 in this case, which is a contradiction of (15) and therefore can be excluded.

6. This implies that, as in step 3, $\underset{\smile}{G}$ needs to be updated upward. This then again leads to an upwards adjustment of $\underset{\smile}{Z}_D$, as in step 4. Eventually this process converges since the solution to the model for $\hat{\theta}_M$ is finite. Let $\underset{\smile}{Z}_D$ and $\underset{\smile}{G}$ be the finite solutions with $\hat{\tau}_D$

the accompanying default time.

7. It holds that,
$$o(\hat{\theta}_M, \hat{\tau}_D) > o(\theta^*_M, \tau^*_D),$$
which contradicts the definition of θ^*_M.

This implies that shareholders have no incentive to deviate from their issuance strategy. Furthermore, the default strategy is optimal and therefore the solution is a Markov perfect equilibrium. □

F. *Restructuring Model*

Assumption 2. *For any $\theta_M \in \mathbb{R}^2_+$ and one-shot first period deviation of θ_M the equity value is non-negative given the default boundary that satisfies the smooth pasting condition. Furthermore, the equity value is non-decreasing in the firm value the next iteration for any Markovian strategy θ_M.*

Theorem 2. *For $\pi < \pi^*(\pi)$ such that Assumption 1 holds there exists a Markov Perfect Equilibrium. In this equilibrium,*

- *The equilibrium issuance strategy θ^*_M maximizes the firm value assuming shareholders default optimally.*
- *The equilibrium default strategy τ^*_D is,*
$$\tau^*_D = \inf\left\{\mathcal{C}, \mathcal{P}\right\},$$
where the default barrier Z^*_D satisfies the smooth pasting condition.
- *The equilibrium debt price equals the expected discounted cash flows of the debt given τ^*_D.*
- *The equilibrium restructuring strategy a^* is given by Proposition 3.*
- *Off-equilibrium, creditors belief that any deviation from the optimal issuance strategy is a one-shot deviation. The off-equilibrium debt price is the expected discounted cash flows of the debt given the optimal default strategy that follows from these beliefs.*

Proof. Take $\theta^*_M = \underset{\theta_M \in \mathbb{R}^2_+}{\arg\sup}\ o(\theta_M, \tau_D)$. From Lemma 4 it follows that if $\eta^{\theta^*_M} \leq \bar{\eta}$ then the optimal firm value is equal to the unlevered firm value 1. Therefore the optimal debt maturity

$\eta^{\theta_M^*} < \infty$ is attained. The optimality and existence of a finite barrier default strategy follows from Lemma 5. The existence of a finite default barrier implies that $Z^{\theta_M^*} < \infty$ is attained.

Because of the time-homogeneity of the model it is sufficient to look at one-shot first period deviations to prove existence of a Markov Perfect Equilibrium. Shareholders deviate from θ_M in the first period with $\hat{Z} > 0$ and $\hat{m}$. Lemma 5 shows that the optimal default strategy is given by (4) with the finite default threshold satisfying the smooth pasting condition. The proof now has several steps:

1. Assume there exists a one-shot deviation $(\hat{Z}, \hat{m})$, with the optimal default strategy given by $\hat{Z}_D$ and $\hat{G} = o(\theta_M^*, a^*, \tau_D)$ and the optimal restructuring strategy by $\hat{a}$, that improves the firm value. Observe that firm value $\hat{f}(Z)$ at $\hat{Z}$ in this case is equal to,

$$\begin{aligned}
\hat{f}(\hat{Z}_0|\hat{Z}_D, \hat{a}, \hat{G}) =& \tilde{\mathbb{E}}_0\left[\int_0^{\tau_{\hat{Z}_D}} e^{-(\delta+\hat{\eta}+\lambda)t}\left(\delta + (\pi - q)Z_t\right)dt\right] \\
&+ \tilde{\mathbb{E}}_0\left[\int_0^{\tau_{\hat{Z}_D}} e^{-(\delta+\hat{\eta}+\lambda)t}\mathbb{1}_{\left\{\hat{G}\geq\rho(1/\hat{\eta})Z_t\hat{G}\right\}}\hat{\eta}\left(o(\theta_M^*, \tau_D^*) - q\rho(1/\hat{\eta})Z_t\right)dt\right] \\
&+ \tilde{\mathbb{E}}_0\left[\int_0^{\tau_{\hat{Z}_D}} e^{-(\delta+\hat{\eta}+\lambda)t}\mathbb{1}_{\left\{\hat{G}<\rho(1/\hat{\eta})Z_t\right\}}\hat{\eta}(1-q)(1-\alpha)dt\right] \\
&+ \tilde{\mathbb{E}}_0\left[\int_0^{\tau_{\hat{Z}_D}} e^{-(\delta+\hat{\eta}+\lambda)t}\lambda\mathfrak{r}(Z_t)dt\right] \\
&+ \tilde{\mathbb{E}}_0\left[e^{-(\delta+\hat{\eta}+\lambda)\tau_{\hat{Z}_D}}(1-q)(1-\alpha)\right] \\
&> o(\theta_M^*, a^*, \tau_D^*).
\end{aligned}$$

2. The mapping K,

$$\begin{aligned}
K(\hat{f})(Z_0|\hat{Z}_D, \hat{G}) =& \tilde{\mathbb{E}}_0\left[\int_0^{\tau_{\hat{Z}_D}} e^{-(\delta+\hat{\eta}+\lambda)t}\left(\delta + (\pi - q)Z_t\right)dt\right] \\
&+ \tilde{\mathbb{E}}_0\left[\int_0^{\tau_{\hat{Z}_D}} e^{-(\delta+\hat{\eta}+\lambda)t}\mathbb{1}_{\left\{\rho(1/\hat{\eta})Z_t\leq\hat{G}\right\}}\hat{\eta}\left(\hat{f}(\hat{Z}) - q\rho(1/\hat{\eta})Z_t\right)dt\right] \\
&+ \tilde{\mathbb{E}}_0\left[\int_0^{\tau_{\hat{Z}_D}} e^{-(\delta+\hat{\eta}+\lambda)t}\mathbb{1}_{\left\{\rho(1/\hat{\eta})Z_t>\hat{G}\right\}}\hat{\eta}(1-q)(1-\alpha)dt\right] \\
&+ \tilde{\mathbb{E}}_0\left[\int_0^{\tau_{\hat{Z}_D}} e^{-(\delta+\hat{\eta}+\lambda)t}\lambda\mathfrak{r}(Z_t)dt\right] \\
&+ \tilde{\mathbb{E}}_0\left[e^{-(\delta+\hat{\eta}+\lambda)\tau_{\hat{Z}_D}}(1-q)(1-\alpha)\right],
\end{aligned}$$

is a contraction mapping just as A (10) in Lemma 2. Furthermore, this mapping is increasing in $\hat{f}(\hat{Z})$. This implies that,

$$\begin{aligned} K^{\infty}(\hat{f})(\hat{Z}|\hat{Z}_D, \hat{G}) &\geq K^{n}(\hat{f})(\hat{Z}|\hat{Z}_D, \hat{G}) \\ &\geq K^{n-1}(\hat{f})(\hat{Z}|\hat{Z}_D, \hat{G}) \\ &\geq K(\hat{f})(\hat{Z}|\hat{Z}_D, \hat{G}) \\ &> o(\theta^*_M, a^*, \tau^*_D) \\ &\geq 1, \end{aligned} \tag{16}$$

where K^n is the mapping K applied n times to $\hat{f}$. The fixed point of this mapping is the solution to the firm value given issuance strategy $\hat{\theta}_M = (\hat{Z}, \hat{m})$ and default strategy $(\hat{Z}_D, \hat{G})$.

3. The intermediate value theorem combined with Lemma A.6 of Hugonnier et al. (2015) implies that there exists an optimal $G \in \left(o(\theta^*_M, a^*, \tau^*_D), \frac{\delta+(\pi-q)\hat{Z}_D}{\delta}\right]$ that solves solves,

$$G = K^{\infty}(\hat{f})(\hat{Z}|G, \hat{Z}_D),$$

Let $\underset{\vee}{G}$ be this fixed point. This implies that,

$$K^{\infty}(\hat{f})(\hat{Z}|\underset{\vee}{G}, \hat{Z}_D) = \underset{\vee}{G} > o(\theta^*_M, a^*, \tau^*_D).$$

4. Assumption 1 says the equity value is increasing in the firm value the next iteration. The equity value given the strategy $\hat{\theta}_M$ dominates the equity value of the one shot deviation given by $(\hat{Z}, \hat{m})$ therefore at the default threshold,

$$\left.\frac{\partial e(Z, \hat{m}|\hat{\theta}_M, \underset{\vee}{a}, \underset{\vee}{G}, \hat{Z}_D)}{\partial Z}\right|_{Z=\hat{Z}_D} \leq 0,$$

where $\underset{\vee}{a}$ is the optimal restructuring strategy that depends on the current issuance strategy and default thresholds. Furthermore, by definition of Z_D^{help} it must be that $\hat{Z}_D < Z_D^{help}$ therefore between $[\hat{Z}_D, Z_D^{help}]$ there exists another default threshold that satisfies the smooth pasting condition $\underset{\vee}{Z}_D$.

5. The firm value is increasing in the default threshold since,

$$\begin{aligned} &\delta + (\pi - q)\hat{Z}_D + \hat{\eta}(1-q)(1-a) > (\delta + \hat{\eta})(1-q)(1-\alpha), \\ &\delta + (\pi - q)\hat{Z}_D + \hat{\eta}\left(f(\hat{Z}|\hat{\theta}_M, \underset{\vee}{a}, \hat{G}, \hat{Z}_D) - q\rho(1/\hat{\eta})\right)\hat{Z}_D > (\delta + \hat{\eta})(1-q)(1-\alpha). \end{aligned}$$

At the default threshold restructuring is always suboptimal because of the issuance costs and therefore the surplus is always zero. For the second case observe that if $(\pi - q - \hat{\eta} q \rho(1/\hat{\eta})) > 0$ the results holds. If $(\pi - q - \hat{\eta} q \rho(1/\hat{\eta})) < 0$, because shareholders never default at maturity for any trajectory of Z_t issuing debt is negative NPV. The firm value is strictly smaller than 1 in this case, which is a contradiction of (16) and therefore can be excluded.

6. This implies that, as in step 3, $\underset{\vee}{G}$ needs to be updated upward. This then again leads to an upwards adjustment of $\underset{\vee}{Z}_D$, as in step 4. Eventually this process converges since the solution to the model for $\hat{\theta}_M$ is finite. Let $\underset{\vee}{Z}_D$ and $\underset{\vee}{G}$ be the finite solutions with $\hat{\tau}_D$ the accompanying default time and $\underset{\vee}{q}$ the optimal restructuring strategy.

7. It holds that,

$$o(\hat{\theta}_M, \hat{\alpha}, \hat{\tau}_D) > o(\theta^*_M, \alpha^*, \tau^*_D),$$

which contradicts the definition of θ^*_M.

This implies that shareholders have no incentive to deviate from their issuance strategy. Furthermore, the default strategy is optimal and therefore the solution is a Markov perfect equilibrium. □

G. *Finite Debt Maturity*

In this section Proposition 2 is proven. This is done by showing that for $q = 0$ and $\rho = 0$ the firm abstains from issuing perpetual debt. Continuity of the firm value in the model parameters then proves the proposition.

For issuance cost q and a firm issuing perpetual debt the optimal firm value is given by,

$$h(q) = \sup_{\theta \in \{(Z,0) | Z \in \mathbb{R}_+\}} o(\theta, \tau_D)\Big|_q,$$

with τ_D the optimal default time. Let Z^0_D be the optimal default threshold of this perpetual debt model.

Define the equity value $\bar{e}(Z, m|\rho, Z_D, q)$ and the firm value $\bar{f}(Z, m|\rho, Z_D, q)$ as a one-shot first-period deviation from $h(q)$,

$$\begin{aligned}
\bar{e}(Z, m|\rho, Z_D, q) =& \tilde{\mathbb{E}}_0 \left[\int_0^{\tau_{Z_D}} e^{-(\delta+\eta)t} \left(\delta - (1-\pi) Z_t + \eta (h(q) - \rho Z_t)^+ \right) dt \right], \\
\bar{f}(Z, m|\rho, Z_D, q) =& \tilde{\mathbb{E}}_0 \left[\int_0^{\tau_{Z_D}} e^{-(\delta+\eta)t} \left(\delta + (\pi - q) Z_t + \eta \mathbb{1}_{\{h(q) \geq \rho Z_t\}} (h(q) - q\rho Z_t) \right) dt \right] \\
&+ \tilde{\mathbb{E}}_0 \left[e^{-(\delta+\eta)\tau_{Z_D}} (1-q)(1-\alpha) \right].
\end{aligned}$$

The optimal default threshold for the equity value $\bar{e}(Z, m|\rho, Z_D, q)$ follows from Lemma 5 and is given by Z_D^m where $m = 1/\eta$ is the maturity of the debt issued.

Lemma 6. *Given the default threshold Z_D^0, initially issuing finite maturity debt for $\rho = 0$ and $q = 0$ increases the firm and equity value,*

$$\bar{f}(Z, \infty|0, Z_D^0, 0) > \bar{f}(Z, m|0, Z_D^0, 0),$$
$$\bar{e}(Z, \infty|0, Z_D^0, 0) > \bar{e}(Z, m|0, Z_D^0, 0).$$

Proof. For the firm value,

$$\bar{f}(Z, \infty|0, Z_D^0, 0) - \bar{f}(Z, m|0, Z_D^0, 0) = \tilde{\mathbb{E}}_0 \left[\int_0^{\tau_{Z_D}} e^{-(\delta+\eta)t} \eta \left(\bar{f}(Z_t, \infty|0, Z_D^0, 0) - h(0) \right) dt \right] < 0.$$

The inequality follows from the fact that,

$$h(0) = \sup_{Z \in \mathbb{R}_+} \bar{f}(Z, \infty|0, Z_D^0, 0).$$

The equity value,

$$\bar{e}(Z, \infty|0, Z_D^0, 0) - \bar{e}(Z, m|0, Z_D^0, 0) = \tilde{\mathbb{E}}_0 \left[\int_0^{\tau_{Z_D}} e^{-(\delta+\eta)t} \eta \left(e(Z_t, \infty|0, Z_D^0, 0) - h(0) \right) dt \right] < 0,$$

follows from the same reasoning. □

Proposition 2. *For small issuance costs q and principals $\rho(m)$ the equilibrium debt maturity m^* from Theorem 1 is finite, that is $m^* < \infty$.*

Proof. For $q = 0$ and $\rho = 0$ the result follows directly from the previous lemma,

$$\bar{e}(Z, \infty|0, Z_D^0, 0) \leq \bar{e}(Z, m|0, Z_D^0, 0) < \bar{e}(Z, m|0, Z_D^m, 0),$$
$$\bar{f}(Z, \infty|0, Z_D^0, 0) \leq \bar{f}(Z, m|0, Z_D^0, 0) < \bar{f}(Z, m|0, Z_D^m, 0).$$

The last inequality follows from the fact that for Z_D^0 the equity value with the finite maturity debt dominates the perpetual debt equity value. This implies that the optimal default threshold $Z_D^m > Z_D^0$. Since the firm value is increasing in the default threshold the inequality for the firm value follows.

This result implies that for $\rho = 0$ and $q = 0$ there is a one-shot deviation that increases the firm value. Therefore, perpetual debt is never issued in equilibrium. Continuity of the firm value in the model parameters (with the optimal default threshold adjusting accordingly), following Lemma A.6 of Hugonnier et al. (2015), implies the result.

$\square$